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Sharp Phase Boundaries for a Lattice Flux Line Model

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We consider a model of nonintersecting flux lines in a rectangular region on the lattice $\mathbb{Z}^d$, where each flux line is a non-isotropic self-avoiding random walk constrained to begin and end on the boundary of the region. The thermodynamic limit is reached through an increasing sequence of such regions. We prove the existence of several distinct phases for this model, corresponding to different regimes for the flux line density—a phase with zero density, a collection of phases with maximal density, and at least one intermediate phase. The locations of the boundaries of these phases are determined exactly for a wide range of parameters. Our results interpolate continuously between previous results on oriented and standard nonoriented self-avoiding random walks.

KEY WORDS: Self-avoiding random walk; phase boundary.

1. INTRODUCTION

We analyse the phase diagram of a statistical model of mutually avoiding, selfavoiding random walks on the d-dimensional hypercubic lattice. The model is defined first in a finite region, with the walks constrained to begin and end on the boundary of the region, and then the thermodynamic limit is constructed. This model arises in two dimensions in connection with

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dimer models and other exactly solvable models,[1–8] and in three dimensions in studies of magnetic flux lines in superconductors[9–13] as well as directed polymer models.[14] Other applications where these random walk models arise include.[15–21] Here we consider a general model of non-isotropic self-avoiding walks in d dimensions (with, in general, different weights for each of the $2d$ lattice directions) which includes all these cases. We prove the existence of several phases in the general case, and we locate their exact boundaries.

We call the random walks *flux lines* because of the analogy with localised magnetic fields in a Type II superconductor—this connection is elaborated below. The model has $2d$ parameters $\{z_{1\pm},..., z_{d\pm}\}$, associated with the $2d$ directions of oriented bonds on the lattice. The statistical weight of a single walk is the product of weights for each lattice bond traversed by the walk—the weight of an ensemble is the product of weights for each walk. We define the grand canonical partition function as the sum of weights over all ensembles of non-intersecting walks that begin and end on the boundary, and the thermodynamic limit is taken through a sequence of rectangular regions. The model considered in ref. 13 is the special case $z_{i-}=0$ for all $i=1,..., d$.

Our main results concern the existence of phase transitions for this model, and the precise location of its phase boundaries. We locate three distinct types of phases: a Meissner phase where there are no flux lines in the bulk; a frozen phase where the flux line density is maximal; and an intermediate flux liquid phase where the density of flux lines is neither zero nor maximal. We also compute all correlation functions in the Meissner and in part of the frozen phases. These results include and extend results obtained in ref. 13.

The phase boundaries of the model are given implicitly by equations involving the direction-dependent weights $\{z_{1\pm},..., z_{d\pm}\}$. To be specific, recall that for isotropic self-avoiding walks in $\mathbb{Z}^d$, the connectivity constant μ gives the exponential rate of growth of the number of walks starting at the origin.[22] We define a generalized connectivity constant $\lambda(z)$ which is the exponential rate of growth of the weighted sum over walks starting at the origin (this is defined precisely in (3.2) below). In the special case $z_{i\pm}=z$ for all $i=1,..., d$, this becomes $\lambda(z)=\mu z$. Then the boundary between the Meissner phase and the flux liquid phase is $\lambda(z)=1$. The model enters a frozen phase when one of the weights (say z_{i+}) becomes significantly larger than the other $2d-1$ weights. In this case there is also a generalized connectivity constant $\lambda^*_{i+}(z)$ for a dual random walk model (this is defined in (3.6) and explained in Sections 5 and 6). Unlike $\lambda(z)$, this can be computed explicitly in terms of the weights. Furthermore, if $z_{i+}z_{i-}\leqslant 1$, then the boundary between this frozen phase and the flux liquid phase occurs at $\lambda^*_{i+}(z)=1$. To summarize, we have the following equations for the boundaries (assuming that $z_{i+}>z_{i-}$ and $z_{i+}z_{i-}\leqslant 1$):

$$\text{Meissner—flux liquid} \qquad \lambda(z)=1 \tag{1.1}$$

$$\text{Frozen—flux liquid} \qquad z_{i+}=1+\sum_{j\neq i}(z_{j+}+z_{j-}) \tag{1.2}$$

The oriented flux line model (with $d=3$) was originally introduced in the physics literature to describe the behavior of a Type II superconductor below its critical temperature, over a range of values of an external applied magnetic field.[9–12] In this model the random walks correspond to localised magnetic fields—the transport of magnetic flux through the superconductor is impossible for weak fields (the Meissner effect) and occurs in localized filaments for strong fields. This behavior gives rise to the triangular lattice of emerging flux lines on the boundary of a sample known as the Abrikosov lattice. The statistical model of random walks is intended to describe fluctuations of the flux lines around the rigid configuration of the Abrikosov lattice.

For the model of magnetic flux lines, the weights are chosen to be $z_{i\pm}=e^{-\beta(\varepsilon\mp h_i)}$ where β is inverse temperature, ε is an energy per unit length of the flux line, and h_i is the ith component of the applied field (which is assumed to be constant). Note that this choice of z always leads to $z_{i+}z_{i-}\leqslant 1$ (since β and ε are positive). In Fig. 1 we show the phase diagram for this model as the parameters h and $T=\beta^{-1}$ vary, where

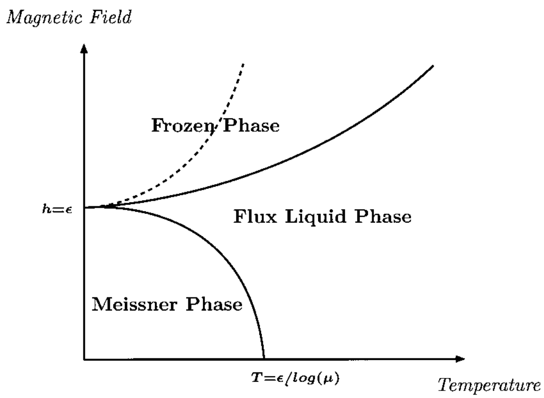

Fig. 1. The phase diagram for mutually avoiding, self-avoiding random walks on the lattice $\mathbb{Z}^d$, in the case which models magnetic flux lines in a Type II superconductor below its critical temperature. The boundary separating the Meissner phase and the flux liquid phase intersects the line $h=0$ at the temperature $T=\varepsilon/\log(\mu)$, where μ is the standard connectivity constant for isotropic self-avoiding random walks on $\mathbb{Z}^d$.

$h = h_1 > 0$ and $h_i = 0$ for $i = 2,..., d$. There are three phases: the Meissner phase where the free energy is zero, and all correlations vanish; the frozen phase (in this case frozen in the $+1$-coordinate direction) where the free energy is $-(h-\varepsilon)/T$; and the flux liquid phase, where the free energy is strictly less than $\min\{0, -(h-\varepsilon)/T\}$. In the sub-region of the frozen phase above the dotted line we can prove in addition that all correlation functions are frozen, i.e., the only contribution to their values comes from the ground state (see Theorem 3.1). We expect that this is true throughout the frozen phase. The equations of the phase boundaries in this case are given as follows:

$$\text{Meissner—flux liquid} \qquad \lambda(z) = 1 \tag{1.3}$$

$$\text{Frozen—flux liquid} \qquad \exp(h/T) = 2d - 2 + \exp(\varepsilon/T) \tag{1.4}$$

If we compare this phase diagram with the corresponding Fig. 6 in ref. 10, we can identify our "flux liquid" phase with the "entangled flux liquid" phase there, and our frozen phase with the "flux lattice" phase. We find good agreement between the qualitative features of our phase diagram and the phase diagrams shown in Fig. 6 in ref. 10, and also in Fig. 1 in ref. 12.

The paper is organized as follows. In Section 2 we define the model. In Section 3 we state our results and discuss their relation to previous work on these questions. Section 4 contains our results on the shape of the boundary of the Meissner phase. Here we use techniques from the theory of self-avoiding random walks, and the position of the boundary is characterized in terms of the connectivity constant for a single self-avoiding walk. In Section 5 we define a duality mapping for flux configurations which allows the frozen phase to be represented as a Meissner phase of a related random walk model. In Section 6 we combine the methods of Sections 4 and 5 to prove our results on the frozen phase.

2. DEFINITION OF THE MODEL

In this paper, we study d-dimensional models of flux lines in a magnetic field. The flux lines are represented by self-avoiding nearest neighbor walks ω on $\mathbb{Z}^d$ with weights

$$z^\omega := \prod_{i=1}^{d} z_{i+}^{N_{i+}(\omega)} z_{i-}^{N_{i-}(\omega)} \tag{2.1}$$

where $N_{i+}(\omega)$ and $N_{i-}(\omega)$ denote the number of steps that the walk ω takes in the positive and negative i-direction, respectively, and $z = (z_{1+}, z_{1-},..., z_{d-})$ denotes a weight vector with $2d$ non-negative entries $z_{i\pm} \geqslant 0$.

While most of our results hold for general weight vectors z with arbitrary nonnegative entries $z_{i\pm}$, we are most interested in the case where the walks represent flux lines in a magnetic field, corresponding to the weight vectors

$$z_{i\pm} = e^{-\beta(\varepsilon \mp h_i)} \qquad i = 1,..., d, \tag{2.2}$$

where $\varepsilon \geqslant 0$ represents an energy per unit length, h_i represents the ith component of a magnetic field $\vec{h}$, and β is the inverse temperature.

As usual, a self-avoiding nearest neighbor walk on $\mathbb{Z}^d$ (SAW for short) is a sequence ω of nearest neighbor points $\omega(0), \omega(1),..., \omega(N) \in \mathbb{Z}^d$ that obey the constraints $\omega(t) \neq \omega(s)$ for $t \neq s$. We write ω: $x \to y$ and say that ω is a walk from x to y if $\omega(0) = x$ and $\omega(N) = y$, and call $|\omega| = N$ the length of the walk $\omega = (\omega(0), \omega(1),..., \omega(N))$. We say that ω is a walk in $\Lambda \subset \mathbb{Z}^d$ and write $\omega \subset \Lambda$ if $\omega(t) \in \Lambda$ for all $t = 0,..., |\omega|$. We also say that ω is a walk from $\partial\Lambda$ to $\partial\Lambda$ (where $\partial\Lambda$ is the set of points in Λ that are adjacent to a point of $\Lambda^c = \mathbb{Z}^d \backslash \Lambda$) if $\omega \subset \Lambda$ and ω: $x \to y$ for some x, $y \in \partial\Lambda$.

Our model is defined in terms of the grand canonical partition function

$$Z(\Lambda) = Z(z; \Lambda) = \sum_{\Omega} \prod_{\omega \in \Omega} z^{\omega} \tag{2.3}$$

where we sum over all sets $\Omega = \{\omega_1,..., \omega_{n(\Omega)}\}$ of non-intersecting self-avoiding random walks $\omega_i \subset \Lambda$ that start and end at the boundary $\partial\Lambda$ of Λ. Notice that there is an upper bound on $n(\Omega)$, the number of walks in Ω, namely $n(\Omega) \leqslant \lfloor |\partial\Lambda|/2 \rfloor$.

The k-point correlation function of the model is defined by restricting the sum in (2.3) to configurations in which the walks contain k given distinct points $x_1,..., x_k$:

$$S_\Lambda(x_1,..., x_k) = Z(\Lambda)^{-1} \sum_{\Omega:\, x_1,..., x_k \in P(\Omega)} \prod_{\omega \in \Omega} z^{\omega} \tag{2.4}$$

where $P(\Omega)$, $\Omega = \{\omega_1,..., \omega_n\}$, denotes the union of all points visited by $\omega_1,..., \omega_n$. We also define the connectivity $\tau_\Lambda(x_1,..., x_k)$ as the probability that all the points $x_1,..., x_k$ are visited by a single random walk ω:

$$\tau_\Lambda(x_1,..., x_k) = Z(\Lambda)^{-1} \sum_{\Omega:\, \exists \omega \in \Omega \text{ s.th. } x_1,..., x_k \in \omega} \prod_{\omega \in \Omega} z^{\omega} \tag{2.5}$$

We define the free energy of the model

$$f = f(z) = -\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z(z; \Lambda), \tag{2.6}$$

the infinite-volume k-point correlation function

$$S(x_1,..., x_k) = \lim_{\Lambda \uparrow \mathbb{Z}^d} S_\Lambda(x_1,..., x_k), \tag{2.7}$$

and the infinite-volume connectivities

$$\tau(x_1,..., x_k) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \tau_\Lambda(x_1,..., x_k), \tag{2.8}$$

where the limits are taken along a sequence of rectangular sets

$$\Lambda = \{x \in \mathbb{Z}^d \mid -L_\mu \leqslant x_\mu \leqslant L_\mu,\ \mu = 1,..., d\} \tag{2.9}$$

Most of our results hold for a sequence satisfying the usual van Hove condition $|\partial\Lambda|/|\Lambda| \to 0$. However for some results we need the following slightly stronger condition:

$$\text{there is } \varepsilon > 0 \text{ such that for all } \Lambda \text{ and all } i, j = 1,..., d,\ L_i \geqslant L_j^\varepsilon \tag{2.10}$$

Remarks. (i) The existence of the limit (2.6) can be shown by an easy subadditivity argument, c.f. ref. 13. We will prove the existence of the limits (2.7) and (2.8) only in the cases described in Propositions 3.2 and 3.3.

(ii) Again by the arguments of ref. 13, it is easy to show that the introduction of a fugacity $y > 0$ per walk leads to the same free energy. This is essentially due to the fact that the scaling in (2.6) involves the volume $|\Lambda|$, while the number of walks in (2.3) grows at most as the size of the boundary of Λ.

(iii) The model considered in ref. 13 is obtained from the more general model considered here by setting $z_{i-} = 0$ for all i.

3. MAIN RESULTS

The theorems in this section will give exact formulas for the phase boundaries of the model (2.3) in terms of certain generalized connectivity constants $\lambda(z)$ and $\lambda_i^*(z)$, which we define below. We also present results on the correlation functions of the model for some ranges of values of the weights z.

To define $\lambda(z)$, we introduce the generating function of all N-step walks that start at the origin $\vec{0} \in \mathbb{Z}^d$:

$$\chi_N(z) = \sum_{\substack{\omega:\, \omega(0)=\vec{0} \\ |\omega|=N}} z^{\omega} \tag{3.1}$$

$\lambda(z)$ is then defined as the limit,

$$\lambda(z) := \lim_{N \to \infty} \chi_N(z)^{1/N} \tag{3.2}$$

(the existence of this limit is easy and will be shown in Section 4). The susceptibility is defined as

$$\chi(z) = \sum_{N=0}^{\infty} \chi_N(z) \tag{3.3}$$

It is easy to show that $\chi(z) < \infty$ if and only if $\lambda(z) < 1$. We also define the symmetrized weights $\bar{z}$ as follows:

$$\bar{z}_{i+} = \bar{z}_{i-} = \sqrt{z_{i+} z_{i-}} \qquad \text{for} \quad i = 1,..., d \tag{3.4}$$

The corresponding susceptibility is denoted $\chi(\bar{z})$. For the weights (2.2), we have $\bar{z}_{i\pm} = e^{-\beta\varepsilon}$ for all i. In this case $\chi(\bar{z})$ is the usual generating function for isotropic self-avoiding walks in d dimensions (see [22, Section 1.3]), which we denote χ_{saw}. Specifically, let C_N be the number of N-step self-avoiding walks starting at the origin. Then

$$\chi(\bar{z}) = \chi_{\text{saw}}(e^{-\beta\varepsilon}) := \sum_{N=0}^{\infty} C_N e^{-N\beta\varepsilon} \tag{3.5}$$

and $\chi(\bar{z}) < \infty$ if and only if $e^{-\beta\varepsilon} < \mu^{-1}$ where μ is the connective constant for self-avoiding walks.

Define

$$\lambda_i^*(z) = \frac{1}{z_{i+}} \left(1 + \sum_{j \neq i} (z_{j+} + z_{j-}) \right) \tag{3.6}$$

As we will show in Section 6, λ_i^* is the connectivity constant of a related walk model describing deviations from the fully packed state, namely the state in which each lattice edge in the positive i-direction is occupied by a self-avoiding walk.

Theorem 3.1. As in (2.2), let $z_{i\pm}=e^{-\beta(\varepsilon\mp h_i)}$ for $i=1,..., d$, and assume that $h_i\geqslant 0$ for all i. Then

(i) $f(z)\leqslant\beta\min\{0, \varepsilon-\max_i h_i\}$.

(ii) $f(z)=0$ if and only if $\lambda(z)\leqslant 1$.

(iii) If $\lambda(z)<1$, both the k-point correlation functions $S(x_1,..., x_k)$ and the connectivities $\tau(x_1,..., x_k)$ are identically zero in the thermodynamic limit.

(iv) $f(z)=\beta(\varepsilon-h_i)$ if and only if $\lambda_i^*(z)\leqslant 1$, i.e., if and only if the following inequality holds: $e^{\beta\varepsilon}+2\sum_{j\neq i}\cosh(\beta h_j)\leqslant e^{\beta h_i}$.

(v) If $\lambda_i^*(z)[1+e^{-\beta\varepsilon}\chi_{\text{saw}}(e^{-\beta\varepsilon})]<1$, then for all $k\geqslant 1$, and all $x_1,..., x_k\in\mathbb{Z}^d$, the limit (2.7) exists and

$$S(x_1,..., x_k)=1, \tag{3.7}$$

and under the condition (2.10) the limit (2.8) exists and

$$\tau(x_1,..., x_k)=\begin{cases}1 & \text{if } x_1,..., x_k \text{ lie on a straight line in the } i\text{th direction}\\ 0 & \text{otherwise}\end{cases} \tag{3.8}$$

The theorem is actually a consequence of the following more general propositions.

Proposition 3.2. Let $z_{j+}\geqslant z_{j-}\geqslant 0$ for all $j=1,..., d$. Then

(i) $f(z)\leqslant -\max\{0, \max_i\log z_{i+}\}$.

(ii) $f(z)=0$ if and only if $\lambda(z)\leqslant 1$.

(iii) If $\lambda(z)<1$, both the k-point correlation functions $S(x_1,..., x_k)$ and the connectivities $\tau(x_1,..., x_k)$ are identically zero in the thermodynamic limit.

For each $i=1,..., d$, define

$$\alpha_i=\max\{1, \sqrt{z_{i-}z_{i+}}\} \tag{3.9}$$

$$\gamma_i(z)=1+\sqrt{z_{i-}z_{i+}}\,\chi(\bar{z}) \tag{3.10}$$

with $\bar{z}$ as defined in (3.4).

Proposition 3.3. Let $z_{j+}\geqslant z_{j-}\geqslant 0$ for all $j=1,..., d$.

(i) If $\lambda_i^*(z)\,\alpha_i\leqslant 1$, then $f(z)=-\log z_{i+}$.

(ii) If $\lambda_i^*(z)>1$, then $f(z)<-\log z_{i+}$.

(iii) If $\lambda_i^*(z)\,\gamma_i(z) < 1$, then for all $k \geqslant 1$, and all $x_1,..., x_k \in \mathbb{Z}^d$, the limit (2.7) exists and

$$S(x_1,..., x_k) = 1 \tag{3.11}$$

(iv) If $\lambda_i^*(z)\,\gamma_i(z) < 1$, and if condition (2.10) holds, then for all $k \geqslant 1$, and all $x_1,..., x_k \in \mathbb{Z}^d$, the limit (2.8) exists and

$$\tau(x_1,..., x_k) = \begin{cases} 1 & \text{if } x_1,..., x_k \text{ lie on a straight line in the } i\text{th direction} \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

Proof of Theorem 3.1, given Propositions 3.2 and 3.3. Since the condition $h_i \geqslant 0$ implies that $z_{i+} \geqslant z_{i-}$, statements (i)–(iii) follow immediately from the corresponding statements in Proposition 3.2.

Since $z_{i+}\,z_{i-} = e^{-2\beta\varepsilon} \leqslant 1$ by the definition (2.2) of $z_{i\pm}$, it follows that $\alpha_i = 1$, and hence statement (iv) follows from Proposition 3.3 (i)–(ii).

Finally, for the weights (2.2), $\gamma_i(z) = 1 + e^{-\beta\varepsilon}\chi_{\text{saw}}(e^{-\beta\varepsilon})$, which reduces statement (v) to the corresponding statements in Proposition 3.3 (iii)–(iv). ∎

Remarks. (i) By Theorem 3.1 (ii) and (iii), the phase which is characterized by $\lambda(z) < 1$ is a phase without any flux lines in the bulk, and hence corresponds to the Meissner phase in a type II superconductor. Indeed, using the methods of ref. 13, one easily shows that the finite volume correlation and connectivity functions (2.4) and (2.5) decay exponentially with the distance of the set $X = \{x_1,..., x_k\}$ from the boundary of Λ if $\lambda(z) < 1$.

(ii) By Theorem 3.1 (v), the region characterized by $\lambda_i^*(z)[1 + e^{-\beta\varepsilon}\chi_{\text{saw}}(e^{-\beta\varepsilon})] < 1$ corresponds to flux lines in the i-direction which are packed as densely as the nonintersecting constraint allows. It therefore corresponds to the so-called frozen phase of a type II superconductor,[12] or the crystal phase described in ref. 10. We expect that the result of Theorem 3.1 (v) holds throughout the phase $\lambda_i^*(z) < 1$, although it appears to be quite difficult to prove this result. As evidence for this conjecture, we point out that the result can be quite easily proved if we use periodic boundary conditions in Λ instead of the boundary condition with sources and sinks.

(iii) By Proposition 3.3 (i) and (ii), the more general model (2.1) can always be driven into a frozen phase by making one of the $2d$ components of the weight vector z large enough. However, only in the case where $z_{i+} z_{i-} \leqslant 1$ can we use Proposition 3.3 (i)–(ii) to find the exact boundary of this frozen phase. Note also that the assumptions of Proposition 3.3 (iii)–(iv) imply that $\chi(\bar{z}) < \infty$ which in turn implies that $z_{i+} z_{i-} < 1$ for all $i = 1,..., d$.

(iv) Obviously, the condition $z_{j-} \leqslant z_{j+}$ in Proposition 3.3 is no restriction of generality, since it can always be achieved by redefining the positive direction of the jth coordinate. The same remark applies to the condition $h_j \geqslant 0$ in Theorem 3.1.

(v) As a special case, the model (2.1) contains the oriented walks model considered in ref. 13. For this model, the methods used in the present paper provide an alternative proof of the results presented in ref. 13. In contrast to the methods used there, our methods here do not rely on the comparison to an exactly solvable model. In addition, our results are more general, since they imply the existence of a frozen phase, with an exact formula for its boundary. Note that this proves the conjecture of Wu and Huang[12] that the exactly solvable model (obtained from our model by an additional factor of $y = -1$ per flux line) has the same phase boundaries as the $y = +1$ model.

(vi) We prove our results on the frozen phase by using a duality transformation which represents the partition function as a sum over configurations of dual flux lines. The dual flux lines in the $+i$-direction are obtained by first representing the configuration of self-avoiding walks (SAW's) by a lattice vector field, then subtracting from this the constant vector field which points in the $+i$-direction, and then finding a new collection of walks which gives rise to this new vector field. These walks are oriented in the $-i$-direction—in fact at least every second step in each walk must be in the $-i$-direction—and so we call them strongly directed walks, or SDW's for short. The fully packed configuration in the $+i$-direction is mapped to the empty configuration under this transformation. We show that the dual model has a Meissner phase, namely a phase where there are no dual flux lines in the bulk, and this gives the frozen phase in the original model. Unlike in the original model, dual flux lines are allowed to share the same site without penalty (at most two dual lines per site). Furthermore two dual lines can share the same edge (only if it is oriented in the $-i$-direction), but with an interaction energy. The interaction is repulsive if $z_{i-} z_{i+} < 1$, and attractive if $z_{i-} z_{i+} > 1$. As noted in Remark ii) above, we can locate the exact boundary of the frozen phase only in the repulsive case.

(vii) The proof of Proposition 3.2 (i), and hence of Theorem 3.1 (i), is immediate, by bounding the partition function from below by 1 (the weight of the empty configuration) and by $(z_{i+})^{|\Lambda|}$ (the weight of a configuration packed maximally in the $+i$-direction). Statements (ii), (iv) of Theorem 3.1 give the conditions under which this bound is saturated. The condition $\lambda_i^*(z) < 1$ can be satisfied for at most one of the coordinate directions $i = 1,..., d$. In particular this implies that a frozen phase cannot occur if the two largest weights are equal. Also the free energy can equal $-\log z_{i+}$ only when the model is frozen in the $+i$-direction.

(viii) We believe that the same results on the phase structure hold for periodic boundary conditions, as long as we insist that all SAWs start and end on the boundary, and then impose periodicity to get loops.

4. THE MEISSNER PHASE

We present the proof of Proposition 3.2 at the end of this section; it uses Lemmas 4.1, 4.2, and 4.4 below. First note that the usual concatenation argument for SAW's shows that

$$\chi_N(z)\,\chi_M(z) \geqslant \chi_{N+M}(z) \qquad \text{for all} \quad N, M \geqslant 0 \tag{4.1}$$

and subadditivity implies the existence of the limit

$$\lambda(z) := \lim_{N\to\infty} \chi_N(z)^{1/N} = \inf_{N\geqslant 1} \chi_N(z)^{1/N} \tag{4.2}$$

Recall the definition of the susceptibility

$$\chi(z) := \sum_{N=0}^{\infty} \chi_N(z) \tag{4.3}$$

and note that, by (4.2), $\chi(z) < \infty$ if and only if $\lambda(z) < 1$.

Lemma 4.1. Suppose $\lambda(z) < 1$. Then $f(z) = 0$.

Proof. For sites $u, v \in \mathbb{Z}^d$, let $G_z(u, v)$ be the generating function of all SAW's (of any length) that start at u and end at v:

$$G_z(u, v) = \sum_{\omega:\, u\to v} z^{\omega} \tag{4.4}$$

It follows that

$$\sum_v G_z(u, v) = \sum_{N=0}^{\infty} \chi_N(z) = \chi(z) \tag{4.5}$$

We bound the partition function by relaxing the condition that the SAW's are non-intersecting, and summing over ordered sets of walks that begin and end on the boundary:

$$\begin{aligned}
Z(z;\Lambda) &\leqslant \sum_k \frac{1}{k!} \sum_{u_1, v_1, \ldots, u_k, v_k \in \partial\Lambda} \prod_{i=1}^{k} G_z(u_i, v_i) \\
&\leqslant \sum_{k=0}^{\lfloor |\partial\Lambda|/2 \rfloor} \frac{1}{k!} \left(\sum_{u, v \in \partial\Lambda} G_z(u, v) \right)^k \\
&\leqslant \exp\left(\sum_{u \in \partial\Lambda} \chi(z) \right) \\
&= \exp(|\partial\Lambda|\, \chi(z))
\end{aligned} \tag{4.6}$$

The $k!$ arises because the sum (2.3) is over *unordered* sets of walks, and hence unordered k-tuples of endpoints (u_i, v_i). Since $\lambda(z) < 1$ we know that $\chi(z) < \infty$, and so the van Hove condition implies that $f(z) = 0$. ∎

Lemma 4.2. Let $z_{j+} \geqslant z_{j-} \geqslant 0$ for all $j = 1,..., d$. Suppose $\lambda(z) > 1$. Then $f(z) < 0$.

Proof. The assumption that $\lambda(z) > 1$ implies that $z_{j+} > 0$ for some j. Assume $j = 1$ for convenience. We say that an N-step SAW ω is a *bridge* if $\omega_1(0) < \omega_1(i) \leqslant \omega_1(N)$ for every $i = 1,..., N$ (here and below we write v_j for the jth coordinate of a lattice site v, and $\omega_j(n)$ for $(\omega(n))_j$). The importance of bridges was realized long ago by Hammersley and Welsh.[25] These bridges were also called "cylinder walks" in ref. 23. For any lattice site v with $v_1 > 0$, let $B_z(0, v)$ be the generating function of all bridges that start at 0 and end at v. Then for all integers $j, k \geqslant 1$,

$$B_z(0, jv)\, B_z(0, kv) \leqslant B_z(0, (j+k)\, v) \tag{4.7}$$

Note that the direction of the inequality in (4.7) is reversed from that in (4.1). In fact this was the original motivation for introducing cylinder walks in ref. 23. The usual subadditivity relations let us define the mass $M[v; z]$ via

$$M[v; z] = \lim_{L \to \infty} \frac{-\log B_z(0, Lv)}{L} = \inf_{L \geqslant 1} \frac{-\log B_z(0, Lv)}{L} \tag{4.8}$$

Furthermore let $B_z^T(0, v)$ be the generating function of all bridges that start at 0, end at v and whose Euclidean distance from the segment $\langle 0, v \rangle$ is at most T. As above we can use subadditivity to define the mass

$$M^T[v; z] = \lim_{L \to \infty} \frac{-\log B_z^T(0, Lv)}{L} = \inf_{L \geqslant 1} \frac{-\log B_z^T(0, Lv)}{L} \tag{4.9}$$

As the following argument shows, $M^T[v; z]$ converges to $M[v; z]$ as $T \to \infty$:

$$\begin{aligned} M[v; z] &= \inf_{L \geqslant 1} \frac{-\log B_z(0, Lv)}{L} \\ &= \inf_{L \geqslant 1} \inf_{T \geqslant 1} \frac{-\log B_z^T(0, Lv)}{L} \\ &= \inf_{T \geqslant 1} \inf_{L \geqslant 1} \frac{-\log B_z^T(0, Lv)}{L} \\ &= \inf_{T \geqslant 1} M^T[v; z] \\ &= \lim_{T \to \infty} M^T[v; z] \end{aligned} \tag{4.10}$$

We claim if $\lambda(z) > 0$ then there exist $T \geqslant 1$ and $y'' \in \mathbb{Z}^d$ with $y''_1 > 0$ such that

$$a := B_z^T(0, y'') > 1 \tag{4.11}$$

To prove this, we first note that, in general, $B_z(0, Lv)$ is not necessarily finite for all L. If it is infinite for some L, then $M[v; z] = -\infty$. However, in this case (4.11) immediately follows. Indeed, assume that $B_z^T(0, y'') \leqslant 1$ for all T and y'' with $y''_1 > 0$. Then $B_z(0, v) \leqslant 1$ for all v with $v_1 > 0$, and hence $M[v; z] \geqslant 0$ for all v with $v_1 > 0$.

Therefore, we may assume without loss of generality that $M[v; z] > -\infty$ and $B_z(0, v) < \infty$ for all v with $v_1 > 0$. We will first show that if $\lambda(z) > 1$, then $M[v; z] < 0$ for some v with $v_1 > 0$. We use the Hammersley–Welsh "unfolding" procedure (see ref. 25 or Section 3.1 of ref. 22). This gives a mapping b from N-step SAW's ω starting at 0 to $N+1$-step bridges $b(\omega)$ starting at 0. This is done by repeatedly reflecting parts of the SAW through hyperplanes of the form $x_1 = \text{constant}$. Each such reflection does not change the directions of steps in the $\pm j$ direction for $j = 2, \ldots, d$, but increases N_{1+} at the expense of N_{1-}. Also, the extra step is in the $+1$ direction. Since $z_{1+} \geqslant z_{1-}$, it follows that $z^{b(\omega)} \geqslant z_{1+} z^{\omega}$ for every SAW ω.

Moreover, the map b is at most $NP_D(N)^2$-to-one, where $P_D(N)$ is the number of partitions of N into distinct integers. It is known that $P^D(N) \leqslant \exp(K\sqrt{N})$ for some constant K.

For each $y \in \mathbb{Z}^d$ and integer $N \geqslant 1$, let $B_{z,N}(0, y)$ be the generating function of the set of N-step bridges that start at 0 and end at y. Then notice that for every $y \in \mathbb{Z}^d$ such that $y_1 > 0$,

$$\sum_{N=1}^{\infty} B_{z,N}(0, y) = B_z(0, y) \tag{4.12}$$

Observe that $B_{z,N}(0, y)$ is non-zero for less than $N(2N+1)^{d-1}$ values of y; let $y^{[N]}$ be a site which maximizes the value of this function. Then, by the argument of the preceding paragraph, we see that

$$\begin{aligned} z_{1+\chi_N}(z) &\leqslant NP_D(N)^2 \sum_y B_{z,N+1}(0, y) \\ &\leqslant N \exp(2K\sqrt{N})\, N(2N+1)^{d-1}\, B_{z,N+1}(0, y^{[N+1]}) \end{aligned} \tag{4.13}$$

Since $\chi_N(z) \geqslant \lambda(z)^N$, and $\lambda(z) > 1$, there exists N and $y' \in \mathbb{Z}^d$, with $y_1' > 0$, such that $B_{z,N+1}(0, y') > 1$. Therefore, by (4.12), $B_z(0, y') > 1$, and hence (by subadditivity) $M[y'; z] < 0$. By (4.10) this implies that $M^T[y'; z] < 0$ for some $T \geqslant 1$, which again gives (4.11).

For each $u \in \mathbb{Z}^d$ and for each integer $k \geqslant 1$ let

$$u^{[k]} = u + ky'', \qquad u^{[0]} = u \tag{4.14}$$

It follows from (4.7) and (4.11) that for any lattice site u, and any integer k,

$$B_z^T(u, u^{[k]}) \geqslant a^k \tag{4.15}$$

Furthermore, let $C_k^T(u)$ denote the sites whose Euclidean distance from the segment $\langle u, u^{[k]} \rangle$ never exceeds T. It is easy to see that if $u_1 = w_1$ and $\|u - w\| \geqslant J$, where $J = 2T\,\|y''\|/y_1''$, then the sets $C_k^T(u)$ and $C_k^T(w)$ are disjoint for any k.

For a lattice vector u we define $u_\perp = 1 + \lfloor (u_2^2 + \cdots + u_d^2)^{1/2} \rfloor$. Recall that our model is defined through an increasing sequence of rectangular regions $\Lambda = [-L_1, L_1] \times \cdots \times [-L_d, L_d]$. It will be convenient now to take $L_1 = ly_1''$ and $L_j = 4ly_\perp''$ for $j = 2,..., d$, where l is an integer. If a lattice site u satisfies the conditions

$$u_1 = -ly_1'', \qquad |u_j| \leqslant ly_\perp'', \qquad j = 2,..., d \tag{4.16}$$

then $u \in \partial\Lambda$, and $u^{[2l]} \in \partial\Lambda$, since $u_1^{[2l]} = ly_1''$ and $|u_j^{[2l]}| \leqslant 3ly_\perp''$. Also for l sufficiently large, $v \in C_{2l}^T(u)$ with $|v_1| \leqslant ly_1''$ implies that $v \in \Lambda$. Hence in this case every bridge appearing in the generating function $B_z^T(u, u^{[2l]})$ lies inside Λ, and so contributes to the partition function $Z(z; \Lambda)$. If u, w are sites satisfying (4.16) and $\|u - w\| > J$, then every term in the product $B_z^T(u, u^{[2l]})\, B_z^T(w, w^{[2l]})$ contributes to the partition function. We can find $(1 + \lfloor 2ly''_\perp / J \rfloor)^{d-1}$ sites satisfying (4.16) which are at least distance J apart; let $S_l(y'')$ denote this collection of sites. Therefore for l sufficiently large we obtain the following lower bound for the partition function:

$$\begin{aligned} Z(z; \Lambda) &\geqslant \prod_{u \in S_l(y'')} \left(B_z^T(u, u^{[2l]}) \right) \\ &\geqslant a^{2l\, |S_l(y'')|} \geqslant a^{2bl^d} \end{aligned} \tag{4.17}$$

for some constant b depending on T and y''. Since $|\Lambda| = l^d (2y''_1)(8y''_\perp)^{d-1}$, this implies that the free energy is negative. ∎

In the course of the proof of Lemma 4.2, we encountered the following result, which may be of independent interest.

Corollary 4.3. Suppose $z_{j\pm} \geqslant 0$ for all j, and $\lambda(z) > 1$. Then there exists a direction $y \in \mathbb{Z}^d$ such that $M[y; z] < 0$.

Remark. In contrast to the case of the isotropic self-avoiding random walk, the direction-dependent mass $M[v; z]$ defined in (4.8) may be negative and finite (for example, the trivial case $z_{1+} = 2$, $z_{1-} = 0$ and $z_{i\pm} = 0$ for all $i > 1$, gives $M[e_1; z] = -\log 2$). This behaviour is examined further in ref. 24.

Lemma 4.4. Suppose $\lambda(z) < 1$. Then both $S(x_1,..., x_k)$ and $\tau(x_1,..., x_k)$ are zero.

Proof. We use the results derived in the proof of Theorem 1 (ii) in ref. 13, where it is shown that $S_\Lambda(x_1,..., x_k)$ is bounded by the same k-point function of the non-interacting model. Although the model in ref. 13 is a special case of the model considered in this paper (namely $z_{j-} = 0$ for $j = 1,..., d$), the argument is identical. The k-point function of the non-interacting model is expressed in turn in terms of the connectivity function of the non-interacting model, which in this case is the connectivity of a single self-avoiding random walk. This decays exponentially with distance from the boundary at the rate $\lambda(z)$, which implies the exponential decay of $S_\Lambda(x_1,..., x_k)$ with distance from the boundary. Since $\tau_\Lambda(x_1,..., x_k) \leqslant S_\Lambda(x_1,..., x_k)$, this yields exponential decay of the connectivities. Therefore both vanish in the limit $\Lambda \uparrow \mathbb{Z}^d$. ∎

Proof of Proposition 3.2. For (i), observe that the partition function is bounded from below by 1, and also by the contribution from the fully packed state in each direction. Part (ii) follows from Lemmas 4.1 and 4.2, and continuity of the free energy. Part (iii) is proved in Lemma 4.4. ▮

5. LATTICE VECTOR FIELDS AND THE DUAL MAPPING

In this section we present the dual representation for a collection of flux lines and flux loops (defined below). It is natural to define the dual representation in this more general setting, and it is also needed for the proof of Proposition 3.3 (i)–(ii). The definition uses lattice vector fields, which we introduce below, together with various classes of objects needed for the construction. The procedure is illustrated in Figs. 2–4, for the case of three SAW's on a two-dimensional lattice.

A *based loop* in Λ is a sequence l of nearest neighbor points $l(0)$, $l(1),..., l(N) \in \Lambda$ with $N=0$ or $N>3$, such that $l(0), l(1),..., l(N-1)$ is a self avoiding walk (if $N>3$), and $l(0)=l(N)$. There is an equivalence relation on based loops; two based loops l and l' are equivalent if they have the same length N, and if there is an integer p such that $l(n)=l'(n+p \textit{ mod } N)$ for all $0 \leqslant n \leqslant N$. A *loop* is an equivalence class of based loops. If $N=0$ we refer to l as a *degenerate loop*; if $N>3$ we refer to l as a *non-degenerate loop*.

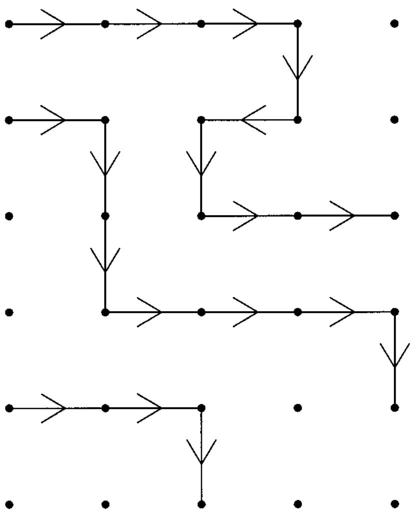

Fig. 2. A configuration of three walks on a finite two-dimensional lattice. The associated flux field has value 1 on each edge with an arrow, and 0 on all other edges.

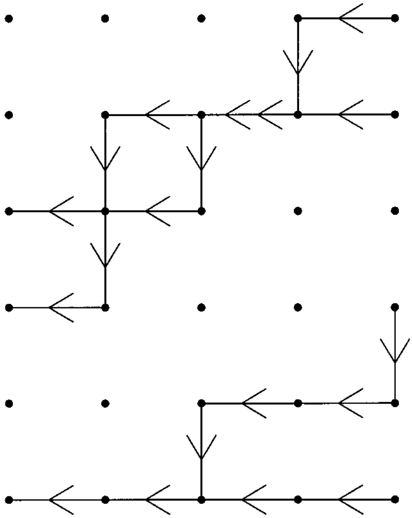

Fig. 3. The dual flux field corresponding to the flux field in Fig. 2, where the frozen direction points to the right. The field has value 2 on the edge with the doubled arrow.

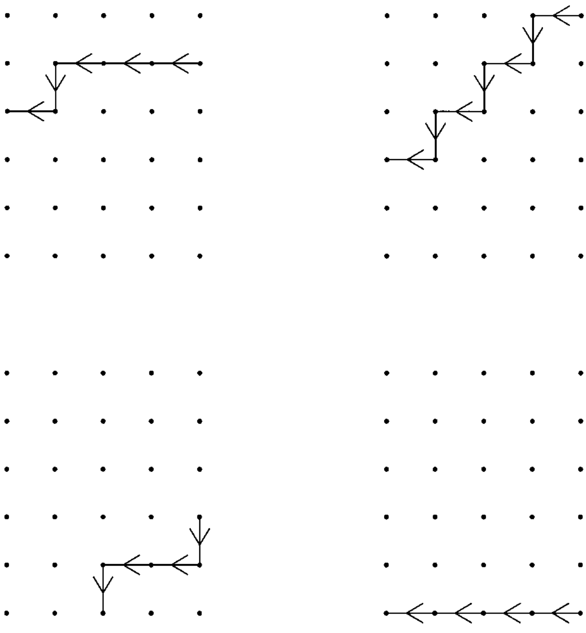

Fig. 4. A dual flux configuration corresponding to the flux field in Fig. 2. The four SDW's in the dual configuration are shown separately. There are four possible ways to choose these walks, depending on the order in which they are constructed.

A *flux configuration* Φ in Λ is a collection of disjoint non-degenerate loops and SAW's, where each SAW begins and ends on the boundary of Λ.

If x, y are nearest neighbor points in $\mathbb{Z}^d$, we denote by $\langle x, y\rangle$ the oriented edge connecting x to y. The collection of all oriented edges with both endpoints in Λ will be denoted $B(\Lambda)$. We will refer to elements of $B(\Lambda)$ as bonds in Λ. Given $b = \langle x, y\rangle$, we will denote the bond with reversed orientation as $r(b)$, so $r(b) = \langle y, x\rangle$.

A *lattice vector field* on Λ is a map $V: B(\Lambda) \to \mathbb{Z}$ satisfying $V(b) = -V(r(b))$ for all $b \in B(\Lambda)$. We introduce a partial ordering on lattice vector fields as follows. Given two lattice vector fields V and W, we write $V \leqslant W$ if $V(b) \leqslant W(b)$ for every bond b with $V(b) > 0$. Note that $V \leqslant W$ implies in particular that the support of V (those bonds where $V(b) \neq 0$) is contained in the support of W.

Given $x \in \Lambda$, let $N(x)$ be the points in Λ which are nearest neighbors of x. A *flux field* is a lattice vector field on Λ satisfying the following conditions for every $x \in \Lambda$: (a) $V(\langle x, y\rangle) \in \{-1, 0, 1\}$ for all $y \in N(x)$; (b) if $x \in \Lambda \backslash \partial\Lambda$, then either $V(\langle x, y\rangle) = 0$ for all $y \in N(x)$, or there are unique points $y_+, y_- \in N(x)$ such that $V(\langle x, y_+\rangle) = V(\langle y_-, x\rangle) = 1$; (c) if $x \in \partial\Lambda$, then there is at most one point $y_+ \in N(x)$ such that $V(\langle x, y_+\rangle) = 1$, and there is at most one point $y_- \in N(x)$ such that $V(\langle y_-, x\rangle) = 1$. Note that in case (c), it is possible for one, both or neither of y_+ and y_- to exist.

Remark. We will refer to y_+ as the outgoing point for V at x, and y_- as the incoming point for V at x. In words, the definition of flux field says that at a point in the interior of Λ there is either one incoming point and one outgoing point, or neither. At the boundary there may be both, only one, or neither.

Let e_j be the unit vector in the jth coordinate direction, for $j = 1, \ldots, d$. So every bond in $B(\Lambda)$ can be written $\langle x, x \pm e_j\rangle$ for some $x \in \Lambda$ and some $j = 1, \ldots, d$. Let E_1 denote the lattice vector field defined by $E_1(\langle x, x \pm e_j\rangle) = \pm\delta_{j,1}$ for all bonds in $B(\Lambda)$, where δ is the Kronecker delta. So E_1 assigns $+1$ to all bonds oriented positively in the first coordinate direction, -1 to all bonds oriented negatively in the first coordinate direction, and zero to all other bonds.

A *dual flux field* W is a lattice vector field on Λ such that $W + E_1$ is a flux field.

A *strongly directed walk* (SDW for short) is a SAW $\theta = \theta(0), \theta(1), \ldots, \theta(N)$ in Λ satisfying (a) $\theta(0), \theta(N) \in \partial\Lambda$, (b) $\theta_1(n+1) - \theta_1(n) \in \{0, -1\}$ for all $0 \leqslant n \leqslant N-1$, and (c) $\theta_1(n+2) - \theta_1(n) \in \{-1, -2\}$ for all $0 \leqslant n \leqslant N-2$ (as before the subscript 1 denotes the first component in $\mathbb{Z}^d$). In words, θ contains no steps in the positive first coordinate direction, and at least one of every two consecutive steps is in the negative first coordinate direction. We will write $|\theta| = N$ for the length of θ.

In Section 6 we will introduce direction dependent weights $\{\zeta_{i\pm}\}$ for these SDW's. Due to the severe constraints on the SDW's, the generating function $\sum_{\theta:\, \theta(0)=0} \zeta^{\theta}$ for a single SDW reduces to the geometric series

$$\sum_{n \geq 0} \Gamma[\zeta_{1-}\Gamma]^n \qquad \text{where} \quad \Gamma = \left(1 + \sum_{j \neq 1} (\zeta_{j+} + \zeta_{j-})\right)$$

The term in square brackets will turn out to be equal to the quantity $\lambda_1^*(z)$ defined in (3.6). This explains why the equation $\lambda_1^*(z) = 1$ gives the boundary between the frozen phase and the flux liquid phase, just as $\lambda(z) = 1$ gives the boundary between the Meissner phase and the flux liquid phase.

Define the following subsets of $\partial\Lambda$:

$$\partial\Lambda_{1+} = \{x \in \Lambda : x + e_1 \notin \Lambda\} \tag{5.1}$$

$$\partial\Lambda_{-1} = \{x \in \Lambda : x - e_1 \notin \Lambda\} \tag{5.2}$$

For $\Lambda = [-L_1, L_1] \times \cdots \times [-L_d, L_d]$, $\partial\Lambda_{1+}$ is the subset of Λ consisting of all points $x = (L_1, x_2,..., x_d)$, and $\partial\Lambda_{1-}$ is the subset $x = (-L_1, x_2,..., x_d)$.

A *maximal strongly directed walk* (MSDW) is a SDW θ such that (a) $\theta(0) \in \partial\Lambda_{1+}$ and $\theta(|\theta|) \in \partial\Lambda_{1-}$, and (b) $\theta(1) = \theta(0) - e_1$, and $\theta(|\theta|) = \theta(|\theta| - 1) - e_1$. So every MSDW contains exactly $2L_1$ steps in the negative first coordinate direction, and its first and last steps are in this direction. The MSDW's will play a role for the dual model analogous to the role played by the bridges in the proof of Lemma 4.2.

There is a straightforward way to associate a unique lattice vector field with a collection of SAW's or loops. First, given a SAW or based loop ϕ in Λ, we define its associated lattice vector field V_ϕ, as follows: if $b = \langle \phi(n), \phi(n \pm 1) \rangle$ for some n, then $V_\phi(b) = \pm 1$; otherwise $V_\phi(b) = 0$. Notice that if ϕ and ψ are equivalent based loops, then $V_\phi = V_\psi$. Second, let Φ be a (not necessarily disjoint) collection of loops and SAW's in Λ. We can choose a representative based loop for each loop; let $\tilde{\Phi}$ denote the resulting collection of SAW's and based loops. The vector field associated with Φ is defined by $V_\Phi = \sum_{\phi \in \tilde{\Phi}} V_\phi$. As a notational aid, we will write W_θ for the vector field associated with a SDW θ, as opposed to the general case of the vector field V_ϕ associated with a SAW or loop.

A *dual flux configuration* Θ is a collection of SDW's on Λ such that the associated lattice vector field $\sum_{\theta \in \Theta} W_\theta$ is a dual flux field. Note that the SDW's in Θ need not be mutually avoiding (though at most two can share the same site).

The following two lemmas are straightforward consequences of the above definitions, but for completeness we include their proofs.

Lemma 5.1. Let Φ be a flux configuration. The lattice vector field V_Φ associated to Φ is a flux field.

Proof. Let V be the vector field associated with Φ. Since the loops and SAW's in Φ are disjoint, $V(b) \in \{-1, 0, 1\}$ for each $b \in B(\Lambda)$. Also if $x \in \phi$ for some $\phi \in \Phi$, then $x = \phi(n)$ for some n, and so if they exist, the incoming and outgoing points for V at x are $y_- = \phi(n-1)$ and $y_+ = \phi(n+1)$. They must exist unless x is an endpoint of ϕ, which happens only if $x \in \partial\Lambda$. If $x \notin \phi$ for any ϕ, then $V(\langle x, y\rangle) = 0$ for all $y \in N(x)$. ∎

Lemma 5.2. Let V be a flux field. There is a unique flux configuration Φ whose associated vector field V_Φ is V.

Proof. We first prove existence by constructing the flux configuration. Let $x \in \Lambda$ such that $V(\langle x, y\rangle) \neq 0$ for some $y \in N(x)$, and let $y_\pm$ denote the incoming and outgoing points of V at x (if they exist). Define $I_+(x) = y_+$ if y_+ exists, and $I_+(x) = x$ otherwise. Similarly define $I_-(x) = y_-$ if y_- exists, and $I_-(x) = x$ otherwise. Note that if $I_+(x) \neq x$, then $I_-(I_+(x)) = x$; similarly if $I_-(x) \neq x$, then $I_+(I_-(x)) = x$.

Define inductively the sequences $\{I_\pm^{[k]}(x)\}$ by $I_\pm^{[0]}(x) = x$, and $I_\pm^{[k+1]}(x) = I_\pm(I_\pm^{[k]}(x))$ for $k \geqslant 0$. Let $k_\pm$ be the largest integer such that $\{I_\pm^{[k]}(x), k = 0,..., k_\pm\}$ is a SAW. So $I_+^{[k_+ + 1]}(x) = I_+^{[j]}(x)$ for some $0 \leqslant j \leqslant k_+$. If $0 < j < k_+$, then $I_+^{[k_+]}(x) = I_+^{[j-1]}(x)$ which contradicts the definition of k_+. Therefore the only possibilities are $j = 0 < k_+$ and $j = k_+ \geqslant 0$. In the first case the sequence $\{I_+^{[j]}(x)\}$, $0 \leqslant j \leqslant k_+ + 1$ is a based loop, starting at x. In the second case the sequence $\{I_+^{[j]}(x)\}$, $0 \leqslant j \leqslant k_+$ is a SAW. Furthermore property (b) of a flux field implies that $I_+^{[k_+]}(x) \in \partial\Lambda$, and hence the SAW ends on $\partial\Lambda$.

The same dichotomy applies to the sequence $\{I_-^{[j]}(x)\}$, giving either $I_-^{[k_- + 1]}(x) = x$ or $I_-^{[k_- + 1]}(x) = I_-^{[k_-]}(x)$. In the first case the sequence $\{I_-^{[j]}(x)\}$ is a based loop starting at x, and it follows that $I_-^{[k_- + 1 - j]}(x) = I_+^{[j]}(x)$ for $j \geqslant 0$. Therefore both sequences give the same based loop, traversed in opposite directions. In this case define the based loop $l = l(0),..., l(N)$ where $N = k_- + 1 = k_+ + 1$, and $l(j) = I_+^{[j]}(x)$ for $0 \leqslant j \leqslant N$.

In the second case $I_-^{[k_- + 1]}(x) = I_-^{[k_-]}(x)$, and the sequence $\{I_-^{[j]}(x)\}$ for $0 \leqslant j \leqslant k_-$ is a SAW which ends on $\partial\Lambda$. Furthermore suppose that $I_-^{[j]}(x) = I_+^{[l]}(x)$ for some $0 < j \leqslant k_-$ and $0 < l \leqslant k_+$. Then $I_-^{[j+l]}(x) = x$, which contradicts the property of being a SAW. Hence the walks $\{I_+^{[j]}(x)\}$, $0 \leqslant j \leqslant k_+$ and $\{I_-^{[j]}(x)\}$, $1 \leqslant j \leqslant k_-$ are disjoint SAW's which end on the boundary. Define a SAW by reversing the second one, and then concatenating them. So the SAW is $\omega = \omega(0),..., \omega(M)$ where $M = k_+ + k_-$, and $\omega(j) = I_-^{[k_- - j]}(x)$ for $0 \leqslant j \leqslant k_- - 1$, and $\omega(j) = I_+^{[j - k_-]}(x)$ for $k_- \leqslant j \leqslant M$.

So far we have associated with every site $x \in \Lambda$ either a loop containing x (it is degenerate if the flux field vanishes at x), or a SAW which begins and ends on $\partial\Lambda$ and contains x. Now we verify that these loops and SAW's are disjoint. Suppose the construction yields a based loop l for the point x, and let y be any other point on this loop. Let l' be the sequence constructed for y. Then $y = I_+^{[j]}(x)$ for some j. Therefore $l'(n) = I_+^{[n]}(y) = I_+^{[n+j]}(x)$ for all n. Hence l and l' are equivalent based loops and so they define the same loop. Similarly, if y belongs to the SAW ω constructed for x, then $y = I_+^{[j]}(x)$ or $y = I_-^{[j]}(x)$ for some j. In either case the SAW constructed for y is equal to ω.

To summarise: we have associated with every site $x \in \Lambda$ a loop containing x, or a SAW which begins and ends on $\partial\Lambda$ and contains x. There is a unique loop or SAW containing every site in Λ, so the loops and SAW's are disjoint. We remove degenerate loops consisting of single points, and let Φ be the resulting collection of non-degenerate loops and SAW's. We have thus shown that Φ is a flux configuration.

It remains to show that the lattice vector field associated with Φ is V. The associated vector field V_Φ takes the value ± 1 only on bonds of the form $\langle x, y\rangle$ where $x = l(n)$, $y = l(n \pm 1)$ for some (based) loop l or $x = \omega(n)$, $y = \omega(n \pm 1)$ for some SAW ω in Φ. In either case $y = I_\pm(x)$, so $V(\langle x, y\rangle) = \pm 1$, and hence $V(\langle x, y\rangle) = V_\Phi(\langle x, y\rangle)$. Therefore V equals the vector field of Φ everywhere.

To prove uniqueness, suppose that Φ' is another flux configuration, and $V = V_\Phi = V_{\Phi'}$. Let $\phi \in \Phi$, and let $x \in \phi$, so $x = \phi(n)$ for some n. If $n < |\phi|$, then $V_\Phi(\langle x, \phi(n+1)\rangle) = 1$, so there must be a unique walk or loop in Φ', say ψ_+, such that $V_{\psi+}(\langle x, \phi(n+1)\rangle) = 1$, and hence $x, \phi(n+1) \in \psi_+$. Similarly if $n > 0$, then $V_\Phi(\langle \phi(n-1), x\rangle) = 1$, so there must be a unique walk or loop in Ω', say ψ_-, such that $V_{\psi-}(\langle \phi(n-1), x\rangle) = 1$, and so $x, \phi(n-1) \in \psi_-$. If $0 < n < |\phi|$, then $x \in \psi_+$ and $x \in \psi_-$. By disjointness we must have $\psi_+ = \psi_-$. If $n+1 < |\phi|$, then again there is a unique walk or loop $\psi_{++} \in \Phi'$ such that $\phi(n+1), \phi(n+2) \in \psi_{++}$, so $\psi_+ = \psi_{++}$. Repeating the argument for all n shows that there is a unique walk or loop $\psi \in \Phi'$ such that $\phi \subset \psi$. Conversely, there is a unique walk or loop $\chi \in \Phi$ such that $\psi \subset \chi$. By disjointness again, we must have $\phi = \chi = \psi$. Hence each loop or walk in Φ also belongs to Φ', and vice versa, so $\Phi = \Phi'$. ∎

Corollary 5.3. There is a 1–1 correspondence between flux configurations and flux fields.

In the next lemma we prove that every flux configuration has a dual representation, namely a (non-unique) collection of SDW's whose associated lattice vector field is the dual flux field of the configuration. The proof is by construction. In order to help visualise the procedure, Figs. 2, 3, and 4 illustrate the steps for a configuration of three SAW's on a two-dimensional lattice.

Lemma 5.4. Let W be a dual flux field.

(a) Let $u \in \Lambda$, and suppose $W(\langle u, v \rangle) > 0$ for some v. Then there is a SDW θ such that $\langle u, v \rangle$ is an edge of θ, $W_\theta \leqslant W$, and $W - W_\theta$ is a dual flux field.

(b) There is a dual flux configuration $\Theta = \{\theta_1,..., \theta_K\}$ whose associated vector field is W, where $K \leqslant 2\,|\partial \Lambda|$. Furthermore, $W_{\theta_{n+1}} \leqslant W_n := W - \sum_{j=1}^{n} W_{\theta_j}$, and $W_{n+1} \leqslant W_n$, for every $n = 0,..., K-1$.

Proof. By definition $W = V - E_1$ for some flux field V. Note this implies that $W(\langle x, x - e_1 \rangle) \in \{0, 1, 2\}$ for all x, and $W(\langle x, x \pm e_j \rangle) \in \{-1, 0, 1\}$ for all $j \neq 1$, and all x. We will use W to define two mappings $J_\pm$ on subsets of $B(\Lambda)$. The map J_+ will be defined on all bonds b with $W(b) > 0$, and it will map this set of bonds into itself. Similarly the map J_- will be defined on bonds b with $W(b) < 0$, and it will map this set of bonds into itself.

Let $b \in B(\Lambda)$, with $W(b) > 0$. First suppose $b = \langle x, x - e_1 \rangle$ for some $x \in \Lambda$. If there is some $k \neq 1$ such that $W(\langle x - e_1, x - e_1 \pm e_k \rangle) = 1$, define $J_+(b) = \langle x - e_1, x - e_1 \pm e_k \rangle$ (there can be at most one such point, since the condition implies $V(\langle x - e_1, x - e_1 \pm e_k \rangle) = 1$, so $x - e_1 \pm e_k$ is the outgoing point for V at $x - e_1$). If not, and $W(\langle x - e_1, x - 2e_1 \rangle) > 0$, define $J_+(b) = \langle x - e_1, x - 2e_1 \rangle$. Otherwise define $J_+(b) = b$. Second, suppose $b = \langle x, y \rangle$, where $y = x \pm e_j$ for some $j \neq 1$. If $W(\langle y, y - e_1 \rangle) > 0$ define $J_+(b) = \langle y, y - e_1 \rangle$. Otherwise define $J_+(b) = b$.

The definition of J_- is similar in case $W(b) < 0$. First suppose $b = \langle x, x + e_1 \rangle$ for some $x \in \Lambda$. If there is some $k \neq 1$ such that $W(\langle x + e_1, x + e_1 \pm e_k \rangle) = -1$, define $J_-(b) = \langle x + e_1, x + e_1 \pm e_k \rangle$ (again there can be at most one such point). If not, and $W(\langle x + e_1, x + 2e_1 \rangle) < 0$, define $J_-(b) = (x + e_1, x + 2e_1\rangle$. Otherwise define $J_-(b) = b$. Second, suppose $b = \langle x, y \rangle$, where $y = x \pm e_j$ for some $j \neq 1$. If $W(\langle y, y + e_1 \rangle) < 0$ define $J_-(b) = \langle y, y + e_1 \rangle$. Otherwise define $J_-(b) = b$.

Suppose $b = \langle x, y \rangle$ and $W(b) > 0$, and $J_+(b) = b$. Then either $y = x - e_1$, or $y = x \pm e_j$ for some $j \neq 1$. If $y = x - e_1$, then either $y - e_1 \notin \Lambda$, which means that $y \in \partial \Lambda_{1-}$, or $W(\langle y, y - e_1 \rangle) = 0$. If $W(\langle y, y - e_1 \rangle) = 0$, then $V(\langle y, y - e_1 \rangle) = -1$, so $y - e_1$ is the incoming point for V at y.

Furthermore, since $W(\langle x, y\rangle) > 0$, so $V(\langle x, y\rangle) \geqslant 0$, and hence x cannot be the outgoing point for V at y. Since $J_+(b) = b$, none of the points $y \pm e_j$ is the outgoing point for V at y. Hence V has no outgoing point at y, which implies again that $y \in \partial\Lambda$. On the other hand, suppose that $y = x \pm e_j$ for some $j \neq 1$. Since $J_+(b) = b$, either $W(\langle y, y - e_1\rangle) = 0$, or $y - e_1 \notin \Lambda$, which implies that $y \in \partial\Lambda_{1-}$. If $W(\langle y, y - e_1\rangle) = 0$ then $y - e_1$ is the incoming point for V at y. But $W(b) > 0$ implies that $V(b) = 1$, so x is the incoming point for V at y, which is a contradiction. Therefore $y \in \partial\Lambda_{1-}$. In all cases it follows from $W(b) > 0$ and $J_+(b) = b$ that $y \in \partial\Lambda$. Similarly the conditions $W(b) < 0$ and $J_-(b) = b$ for the bond $b = \langle x, y\rangle$ imply that $y \in \partial\Lambda$.

To prove part (a), we will use the maps $J_\pm$ to build a SDW containing u. We will then subtract from W the associated vector field of this SDW, and show that the resulting vector field is again a dual flux field.

Let b be the bond $\langle u, v\rangle$, see statement (a). By assumption $W(b) > 0$. Note that by definition if $W(b) > 0$ then $W(J_+(b)) > 0$, and if $W(b) < 0$ then $W(J_-(b)) < 0$. Define by induction the sequence $J_+^{[n]}(b) = J_+(J_+^{[n-1]}(b))$ for $n \geqslant 1$, and $J_+^{[0]}(b) = b$. Write $J_+^{[n]}(b) = \langle x_n, x_{n+1}\rangle$, where $x_0 = u$ and $x_1 = v$. Note by the definition of J_+ that at least every second step in the walk $(x_0, x_1, x_2,...)$ is in the negative first coordinate direction. Let j_+ be the smallest integer for which $J_+^{[j_+ +1]}(b) = J_+^{[j_+]}(b)$. Then the sequence $x_0, x_1,..., x_{j_+ +1}$ is a SDW which starts at $x_0 \in \Lambda$ and ends at $x_{j_+ +1} \in \partial\Lambda$. Similarly define by induction the sequence $J_-^{[n]}(r(b)) = J_-(J_-^{[n-1]}(r(b)))$ for $n \geqslant 1$, and $J_-^{[0]}(r(b)) = r(b)$ (recall that $r(b)$ is the reverse of the oriented bond b). Write $J_-^{[n]}(r(b)) = \langle x_{-n+1}, x_{-n}\rangle$, for $n \geqslant 0$. Let j_- be the smallest integer for which $J_-^{[J_- -1]}(r(b)) = J_-^{[j_-]}(r(b))$. Then the sequence $x_{-j_-},..., x_{-1}, x_0, x_1$ is a SDW which starts at $x_{-j_-} \in \partial\Lambda$ and ends at $x_1 \in \Lambda$. By removing x_1 from the first and then concatenating the walks we obtain a SDW θ of length $j_- + j_+ + 1$ which begins at x_{-j_-} and ends at $x_{j_+ +1}$.

Let W_θ be the vector field associated with θ. Note that $W_\theta(b) > 0$ implies that $W(b) > 0$, and therefore $W_\theta \leqslant W$. Since θ is a SDW which begins and ends on $\partial\Lambda$, it follows that W_θ is a flux field. Let $W_1 = W - W_\theta = V - W_\theta - E_1$. We wish to show that $V_1 = V - W_\theta$ is also a flux field, as this will imply that W_1 is a dual flux field. If $x \notin \theta$ then $V_1(\langle x, y\rangle) = V(\langle x, y\rangle)$ for all $y \in N(x)$, so the conditions for a flux field are satisfied at x. Suppose $x \in \theta$. If $V(\langle x, y\rangle) = 0$ for all $y \in N(x)$, then $V_1(\langle x, y\rangle) = -W_\theta(\langle x, y\rangle)$ for all $y \in N(x)$, and therefore V_1 satisfies the conditions for a flux field at x since $-W_\theta$ is a flux field.

Assume now that $x \in \Lambda \backslash \partial\Lambda$ and let $y_\pm$ be the outgoing and incoming points for V at x. Similarly let $z_\pm$ be the outgoing and incoming points for W_θ at x. Note first that if $y_\pm = z_\pm$ then $V_1(\langle x, y\rangle) = 0$ for all $y \in N(x)$. If $y_+ = z_+$, and $y_- \neq z_-$, then y_- and z_- are respectively the incoming and outgoing points for V_1 at x. If $y_- = z_-$, and $y_+ \neq z_+$, then z_+ and y_+ are respectively the incoming and outgoing points for V_1 at x. The remaining case is $y_- \neq z_-$ and $y_+ \neq z_+$. We will show next that this cannot happen.

Suppose that $y_- \neq z_-$ and $y_+ \neq z_+$. If $z_- = x \pm e_k$ for some $k \neq 1$, then $W(\langle z_-, x\rangle) = 1$ (since $W_\theta \leqslant W$), which implies $V(\langle z_-, x\rangle) = 1$, and hence $y_- = z_-$. Similarly if $z_+ = x \pm e_k$ for some $k \neq 1$, then $y_+ = z_+$. Hence it reduces to the case $z_- = x + e_1$ and $z_+ = x - e_1$. Since $W_\theta \leqslant W$, it must be true that $W(\langle x+e_1, x\rangle) > 0$ and $W(\langle x, x-e_1\rangle) > 0$. This in turn implies that $V(\langle x+e_1, x\rangle) \neq E_1(\langle x+e_1, x\rangle) = -1$, and also $V(\langle x, x-e_1\rangle) \neq -1$. Therefore $y_- \neq x - e_1$ and $y_+ \neq x + e_1$. Hence $y_- = x \pm e_k$ and $y_+ = x \pm e_l$ for some $k, l \neq 1$. Therefore W has two incoming points $x + e_1$, y_- and two outgoing points $x - e_1$, y_+ at x. But in this case the definition of the maps $J_\pm$ gives

$$J_+(\langle y_-, x\rangle) = \langle x, x-e_1\rangle, \qquad J_+(\langle x+e_1, x\rangle) = \langle x, y_+\rangle \tag{5.3}$$

$$J_-(\langle y_+, x\rangle) = \langle x, x+e_1\rangle, \qquad J_-(\langle x-e_1, x\rangle) = \langle x, y_-\rangle \tag{5.4}$$

In no case does this produce the sequence $\dots, x+e_1, x, x-e_1, \dots$ for θ. Therefore it cannot happen that both $y_+ \neq z_+$ and $y_- \neq z_-$. So for all points $x \in \Lambda \backslash \partial\Lambda$ we have shown that V_1 satisfies the conditions for a flux field at x.

If $x \in \partial\Lambda$ it may happen that some of the points $y_\pm$, $z_\pm$ are absent. We can assume at least one of each pair exists. If all of them exist the previous argument applies. Suppose that y_- is missing. Then if z_- exists, it must equal $x \pm e_1$, since otherwise $y_- = z_-$ would not be missing, and since θ is a SDW it must be $x + e_1$. Therefore y_+ cannot be $x + e_1$ (since this would mean $W(\langle x+e_1, x\rangle) = 0$, which contradicts $W_\theta \leqslant W$). Therefore z_+ must exist and equal y_+ (since $J_+(\langle x+e_1, x\rangle) = \langle x, y_-\rangle$), in which case V_1 has no incoming point at x, and $x + e_1$ is the outgoing point. If in addition z_- does not exist, then either $y_+ = z_+$, so V_1 is zero at x, or else z_+ is the incoming point for V_1 and y_+ is the outgoing point. Similar reasoning applies if y_+ is missing. So in all cases there is at most one incoming point and one outgoing point for V_1 at x. This completes the proof that V_1 is a flux field, and hence that $W - W_\theta$ is a dual flux field.

To prove part (b), write $\theta_1 = \theta$, so $W_1 = W - W_{\theta_1}$. Since W_1 is a dual flux field, we can repeat the argument above and construct another SDW θ_2, define its associated vector field V_{θ_2}, and obtain a new dual flux field

$$W_2 = W_1 - W_{\theta_2}$$

for which $W_{\theta_2} \leqslant W_1$. In this way we construct a sequence of SDW's $\{\theta_1, \theta_2, \dots\}$ and the corresponding sequence of dual flux fields $\{W_1, W_2, \dots\}$, satisfying $W_{\theta_{n+1}} \leqslant W_n$. The construction can be continued as long as there is a bond b with $W_n(b) > 0$. Define

$$\|W\|_+ = \sum_{b \in \mathrm{pos}(W)} W(b) \qquad \text{where} \quad \mathrm{pos}(W) = \{b \in B(\Lambda) : W(b) > 0\} \tag{5.5}$$

We claim that $W_{n+1} \leqslant W_n$ for every n. Indeed, let b be a bond with $W_{n+1}(b) > 0$. Suppose that $W_n(b) < W_{n+1}(b)$. Then $W_{\theta_{n+1}}(b) = W_n(b) - W_{n+1}(b) < 0$, which implies $W_n(r(b)) \geqslant W_{\theta_{n+1}}(r(b)) > 0$. Therefore $W_n(b) < 0$, which implies that $|W_{\theta_{n+1}}(b)| = |W_n(b) - W_{n+1}(b)| \geqslant 2$. This is impossible, so we conclude that $W_n(b) \geqslant W_{n+1}(b)$.

Furthermore, since $W_{n+1} \leqslant W_n$, and also $W_{\theta_{n+1}} \leqslant W_n$, the equality $W_n = W_{n+1} + W_{\theta_{n+1}}$ implies that

$$\|W_n\|_+ = \|W_{n+1}\|_+ + \|W_{\theta_{n+1}}\|_+$$

Hence $\|W_n\|_+ > \|W_{n+1}\|_+ \geqslant 0$ for all n, and hence there is an integer K such that $\|W_K\|_+ = 0$, which implies $W_K = 0$. This gives the representation

$$W = \sum_{n=1}^{K} W_{\theta_n} \tag{5.6}$$

This proves that the collection of SDW's $\Theta = \{\theta_1, ..., \theta_K\}$ is a dual flux configuration, and that its associated vector field is W.

Since $W(b) \leqslant 2$ for all $b \in B(\Lambda)$, a point in Λ cannot belong to more than four distinct SDW's. To see this, consider for any $x \in \Lambda$ the quantity $\|W\|_x = \sum_{y \in N(x) \cap \Lambda} |W(\langle x, y \rangle)|$. Then $\|W\|_x \leqslant 4$, and $\|W\|_x = \sum_{n=1}^K \|W_{\theta_n}\|_x$ (this follows from the identity $|W(b)| = \sum_{n=1}^K |W_{\theta_n}(b)|$ for all $b \in B$). If $x \in \Lambda \backslash \partial\Lambda$, the SDW's $\{\theta_n\}$ cannot begin or end at x, and $\|W_{\theta_n}\|_x \in \{0, 2\}$ for all n, so therefore x belongs to at most two of these walks. If $x \in \partial\Lambda$, the same argument leads to the conclusion that x cannot belong to more than four walks from the collection $\{\theta_1, ..., \theta_K\}$. Combining this with the fact that every SDW contains at least two points on the boundary shows that $K \leqslant 2\,|\partial\Lambda|$. ∎

Remarks. (1) It is easy to show in fact that any point on $\partial\Lambda$ can belong to at most two different SDW's from the collection $\{\theta_1, ..., \theta_K\}$.

(2) In general there are many dual flux configurations with the same dual flux field. The next result describes a particular case where the configuration is unique.

Lemma 5.5. Let Ψ be a collection of disjoint MSDW's. Then Ψ is a dual flux configuration, that is its associated vector field W is a dual flux field. Furthermore, $W+E_1$ is the vector field associated to a flux configuration which contains no loops, and Ψ is the unique collection of disjoint MSDW's with associated vector field W.

Proof. We will show first that $V=W+E_1$ is a flux field. First, suppose $x\notin\psi$ for all $\psi\in\Psi$, so $W(\langle x,y\rangle)=0$ for all $y\in N(x)$. Then $x-e_1$ is the incoming point for V at x, and $x+e_1$ is the outgoing point.

Now suppose $x\in\psi$ for some $\psi\in\Psi$. By assumption ψ is unique. So $x=\psi(n)$ for some n; assume at first that $0<n<|\psi|$. Since ψ is a SDW, either $\psi(n-1)=x+e_1$ or $\psi(n+1)=x-e_1$, or both. If $\psi(n-1)\neq x+e_1$, then $\psi(n+1)=x-e_1$, so $V(\langle x,x-e_1\rangle)=0$. Hence $x+e_1$ is the outgoing point and $\psi(n-1)$ is the incoming point for V at x. Similarly if $\psi(n+1)\neq x-e_1$, then $\psi(n+1)$ is the outgoing point and $x-e_1$ is the incoming point. If both are equal then $V(\langle x,y\rangle)=0$ for all $y\in N(x)$. If $n=0$ then, since ψ is a MSDW, $\psi(1)=x-e_1$ and $x+e_1\notin\Lambda$. Hence $V(\langle x,y\rangle)=0$ for all $y\in N(x)$. Similarly if $n=|\psi|$ then $V(\langle x,y\rangle)=0$ for all $y\in N(x)$. Therefore V is a flux field, so Ψ is a dual flux configuration.

It follows from Lemma 5.2 that V is the vector field associated to a flux configuration Ω. We will show that there are no loops in Ω. Since $V(\langle x,x+e_1\rangle)\in\{0,1\}$ for all $x\in\Lambda$, any loop in Ω must contain (at least) three consecutive points of the form $x\pm e_j$, x, $x\pm e_k$ for some $j,k\neq 1$. So at the point x, $x\pm e_k$ is the outgoing point for V and $x\pm e_j$ is the incoming point. But as the above construction shows, either the outgoing point is $x+e_1$ or the incoming point is $x-e_1$, or both; therefore there cannot be such a sequence. Hence Ω contains no loops.

It remains to show that Ψ is the unique collection of disjoint MSDW's with associated vector field W. To this end, note that a collection of disjoint MSDW's is also a flux configuration. Therefore by Lemma 5.2 two different collections cannot share the same vector field. ∎

Remark. Although we will not need it, we note that there is a simple characterization of the allowed SDW's in a dual flux configuration. Namely a collection of SDW's $\{\theta_1,\dots,\theta_K\}$ is a dual flux configuration if and only if (i) each site $x\in\Lambda$ belongs to at most two walks, and (ii) if x belongs to two different walks, then $x+e_1$ is the incoming site for at least one of the walks (assuming that $x+e_1\in\Lambda$), and $x-e_1$ is the outgoing site for at least one of the walks (assuming that $x-e_1\in\Lambda$).

6. THE FROZEN PHASE

We will use the dual mapping developed in Section 5 to prove Proposition 3.3 for the case when the frozen phase is directed in the positive first coordinate direction. The result for other directions is proved in the same way. Recall that $N_{i\pm}(\omega)$ is the number of bonds traversed by the SAW ω in the $\pm i$ direction. We will use the same notation $N_{i\pm}(l)$ for the number of bonds traversed by the loop l in the $\pm i$ direction. For any flux configuration Φ we now define

$$N_{i\pm}(\Phi) = \sum_{\phi \in \Phi} N_{i\pm}(\phi) \tag{6.1}$$

We also define $N_1(\Lambda)$ to be the number of points $x \in \Lambda$ such that $x + e_1$ is also in Λ, that is $N_1(\Lambda) = |\Lambda \backslash \partial\Lambda_{1+}|$.

If V is a lattice vector field on Λ, define for $i = 1,..., d$

$$\mathrm{supp}_{i\pm}(V) = \{x \in \Lambda : x \pm e_i \in \Lambda,\ V(\langle x, x \pm e_i \rangle) > 0\}$$

$$S_{i\pm}(V) = \sum_{x \in \mathrm{supp}_{i\pm}(V)} V(\langle x, x \pm e_i \rangle) \tag{6.2}$$

and also

$$\mathrm{supp}(V) = \bigcup_{i=1}^{d} (\mathrm{supp}_{i+}(V) \cup \mathrm{supp}_{i-}(V)) \tag{6.3}$$

If V and W are lattice vector fields with $V \leqslant W$, it follows that for $i = 1,..., d$

$$S_{i\pm}(W) = S_{i\pm}(V) + S_{i\pm}(W - V) \tag{6.4}$$

If Φ is a flux configuration with associated flux field V_Φ, then

$$N_{i\pm}(\Phi) = S_{i\pm}(V_\Phi) \qquad \text{for} \quad i = 1,..., d \tag{6.5}$$

Using (2.3), (2.1), (6.1) and (6.5) the partition function can be written as

$$\begin{aligned} Z(\Lambda) &= \sum_{\Omega} \prod_{i=1}^{d} z_{i+}{}^{N_i+(\Omega)} z_{i-}{}^{N_{i-}(\Omega)} \\ &= \sum_{\Omega} \prod_{i=1}^{d} z_{i+}{}^{S_i+(V_\Omega)} z_{i-}{}^{S_{i-}(V_\Omega)} \end{aligned} \tag{6.6}$$

where the sum runs over collections of disjoint SAW's which begin and end on $\partial\Lambda$. For a flux configuration Φ, let V_Φ be its associated flux field and

$W_\Phi = V_\Phi - E_1$ the dual flux field. Then clearly $S_{1+}(W_\Phi)=0$, and $S_{i\pm}(W_\Phi) = S_{i\pm}(V_\Phi)$ for $i=2,\ldots,d$. Also $S_{1-}(W_\Phi) = S_{1-}(V_\Phi) + N_1(\Lambda) - S_{1+}(V_\Phi)$. Applying these relations to (6.6) gives

$$(z_{1+})^{-N_1(\Lambda)}\, Z(\Lambda) = \sum_{\Omega} (z_{1+} z_{1-})^{S_{1-}(V_\Omega)}\, (z_{1+}^{-1})^{\;S_{1-}(W_\Omega)} \times \prod_{i=2}^{d} z_{i+}{}^{S_{i+}(W_\Omega)} z_{i-}{}^{S_{i-}(W_\Omega)} \tag{6.7}$$

It will be convenient to define new weights $\zeta = (\zeta_{1+}, \zeta_{1-}, \ldots, \zeta_{d-})$ for the dual model as follows:[5]

$$\zeta_{1+} = 0, \qquad \zeta_{1-} = z_{1+}^{-1}, \qquad \zeta_{i\pm} = z_{i\pm} \qquad \text{for} \quad i=2,\ldots,d \tag{6.8}$$

Since W_Ω is a dual flux field, it follows from Lemma 5.4(b) that there are SDW's $\{\theta_j\}$ such that $W_\Omega = \sum_j V_{\theta_j}$, and such that $V_{\theta_{n+1}} \leqslant W_\Phi - \sum_{j=1}^{n} V_{\theta_j}$ for all n. Therefore by iterating (6.4), for $i=1,\ldots,d$, we deduce that $S_{i\pm}(W_\Omega) = \sum_j S_{i\pm}(V_{\theta_j})$. For any SDW θ we define

$$\zeta^\theta = \zeta_{1-}^{S_{1-}(V_\theta)} \prod_{i=2}^{d} \zeta_{i+}^{S_{i+}(V_\theta)} \zeta_{i-}^{S_{i-}(V_\theta)} \tag{6.9}$$

Then (6.7) can be written as follows:

$$(z_{1+})^{-N_1(\Lambda)}\, Z(\Lambda) = \sum_{\Omega} (z_{1+} z_{1-})^{S_{1-}(V_\Omega)} \prod_j \zeta^{\theta_j} \tag{6.10}$$

Note that $S_{1-}(V_\Omega)$ is the number of doubly occupied dual bonds, and hence the factor $(z_{1+} z_{1-})^{S_{1-}(V_\Omega)}$ is the interaction between dual walks which share the same bond. As noted in Section 3, the interaction is repulsive if $z_{1+} z_{1-} < 1$ and attractive if $z_{1+} z_{1-} > 1$. In addition, if for some $x \in \Lambda$, $V_\Omega(\langle x, x-e_1\rangle) = 1$, then also $W_\Omega(\langle x, x-e_1\rangle) = 2$. Hence

$$S_{1-}(V_\Omega) \leqslant \tfrac{1}{2} S_{1-}(W_\Omega) \tag{6.11}$$

We now claim that the ζ-susceptibility of a single SDW in $\mathbb{Z}^d$ is given by

$$\sum_{\theta:\, \theta(0)=0} \zeta^\theta = \sum_{n\geqslant 0} \Gamma[\zeta_{1-}\Gamma]^n \qquad \text{where} \quad \Gamma = \left(1 + \sum_{j\neq 1} (\zeta_{j+} + \zeta_{j-})\right) \tag{6.12}$$

[5] We note that the weight ζ_{1+} actually never appears in any formula in this paper. We have set it to zero to indicate that SDW's do not take steps in the positive 1-direction.

Indeed, let

$$S_k = \sum_{n=0}^{\infty} \sum_{\theta:\, \theta(0)=0,\, |\theta|=n,\, \theta_1(n)=k} \zeta^\theta$$

so that

$$\sum_{\theta:\, \theta(0)=0} \zeta^\theta = \sum_{k=0}^{\infty} S_k$$

To prove (6.12), we now want to show by induction that

$$S_k = \Gamma(\zeta_{1-}\Gamma)^k$$

Indeed, $S_0 = \Gamma$. To obtain the inductive relation

$$S_k = S_{k-1}\left[\zeta_{1-}\left(1 + \sum_{j\neq 1} (\zeta_{j+} + \zeta_{j-})\right)\right]$$

for $k \geqslant 1$, we distinguish the two cases $\theta_1(n-1) = k-1$ and $\theta_1(n-1) = k$, and observe that in the latter case, necessarily $\theta_1(n-2) = k-1$.

Proof of Proposition 3.3(i). If $\lambda_1^*(z)\,\alpha_1 \leqslant 1$ then $f(z) = -\log(z_{1+})$.

Proof. By keeping only the contribution from the configuration which is maximally packed in the $+1$-coordinate direction, we get

$$Z(\Lambda) \geqslant (z_{1+})^{N_1(\Lambda)} \tag{6.13}$$

Since $N_1(\Lambda)/|\Lambda| \to 1$ as $\Lambda \uparrow \mathbb{Z}^d$, this proves that $f(z) \leqslant -\log(z_{1+})$ (note that this relation was established previously in the course of proving Proposition 3.2 at the end of Section 4).

For the lower bound we distinguish two cases: (i) $z_{1-} z_{1+} \leqslant 1$, and (ii) $z_{1-} z_{1+} > 1$.

(i) $z_{1-} z_{1+} \leqslant 1$: In this case $\lambda_1^*(z)\,\alpha_1 = \lambda_1^*(z)$, and we obtain the following upper bound from (6.10):

$$(z_{1+})^{-N_1(\Lambda)}\, Z(\Lambda) \leqslant \sum_{\Omega} \prod_j \zeta^{\theta_j} \tag{6.14}$$

We increase the right side of (6.14) by summing over all collections of SDW's (not necessarily disjoint) which begin and end on $\partial\Lambda$. Writing this first as a sum over ordered sets of SDW's gives

$$
\begin{aligned}
(z_{1+})^{-N_1(\Lambda)} Z(\Lambda) &\leqslant \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\theta_1,\ldots,\theta_n} \zeta^{\theta_1}\cdots\zeta^{\theta_n} \\
&= \exp\left[\sum_{\theta} \zeta^{\theta}\right]
\end{aligned}
\tag{6.15}
$$

From (6.12) and the observation that $\zeta_{1-}\Gamma = \lambda_1^*(z)$, see (3.6) and (6.8), it follows that for $\lambda_1^*(z) < 1$ there is $C = C(z) < \infty$ such that for any $x \in \Lambda$

$$
\sum_{\theta:\, \theta(0)=x} \zeta^{\theta} \leqslant C \tag{6.16}
$$

Since all walks begin on $\partial\Lambda$, we obtain from (6.15) and (6.16) that

$$
(z_{1+})^{-N_1(\Lambda)} Z(\Lambda) \leqslant \exp[C\, |\partial\Lambda|] \tag{6.17}
$$

For $\lambda_1^*(z) < 1$, (6.17) implies that $f(z) \geqslant -\log(z_{1+})$. Hence $f(z) = -\log(z_{1+})$ for $\lambda_1^*(z) < 1$; by continuity this extends to $\lambda_1^*(z) = 1$.

(ii) $z_{1-}z_{1+} > 1$: Define a new weight vector ζ' by $\zeta'_{1-} = \sqrt{z_{1-}z_{1+}^{-1}}$, $\zeta'_{1+} = 0$, and $\zeta'_{i\pm} = \zeta_{i\pm}$ for $i = 2,\ldots, d$. Then from (6.10) and (6.11) we obtain

$$
(z_{1+})^{-N_1(\Lambda)} Z(\Lambda) \leqslant \sum_{\Omega} \prod_{j} (\zeta')^{\theta_j} \tag{6.18}
$$

This expression is estimated in the same way as (6.14), namely by summing over all SDW's and using the one-walk susceptibility, so the condition $\lambda_1^*(z)\,\alpha_1 = \zeta'_{1-}(1 + \sum_{i=2}^{d} [\zeta'_{i+} + \zeta'_{i-}]) \leqslant 1$ now guarantees that $f(z) = -\log(z_{1+})$. ▮

Proof of Proposition 3.3(ii). If $\lambda_1^*(z) > 1$ then $f(z) < -\log(z_{1+})$.

Proof. The result will follow from a lower bound for $(z_{1+})^{-N_1(\Lambda)} Z(\Lambda)$, which we will derive using an argument similar to that used for the lower bound of $Z(\Lambda)$ in the Meissner phase. Namely we will use the dual representation (6.10) and fill up Λ with disjoint tubes, each containing one MSDW (see Section 5, before Lemma 5.1, for the definition of MSDW). Then (6.10) will be bounded from below by the product of single-walk partition functions for each tube, and provided that $\lambda_1^*(z)>1$ we will show that this product grows exponentially with $|\Lambda|$.

Let $G_N(\zeta)$ be the generating function of all SDW's that start at the origin and take exactly N steps in the -1 direction, and whose first and last steps are in the -1 direction. Also let $H_\zeta(u, v)$ be the generating function of all SDW's that start at u and end at v, for any lattice sites u and v, and again whose first and last steps are in the -1 direction. For $N\geqslant 1$

$$G_N(\zeta)=(\zeta_{1-})^N\,\Gamma^{N-1}=\zeta_{1-}(\lambda_1^*(z))^{N-1}=\sum_{v:\,v_1=-N} H_\zeta(0, v) \quad (6.19)$$

Let $v^{[N]}$ be a site which maximizes $H_\zeta(0, v)$ with the condition $v_1^{[N]}=-N$. Then we have the bound

$$G_N(\zeta)\leqslant(2N-1)^{d-1}\,H_\zeta(0, v^{[N]}) \quad (6.20)$$

Since $\lambda_1^*(z)>1$ there exist N_0 and $v^{[N_0]}$, with $v_1^{[N_0]}=-N_0$, such that

$$h:=H_\zeta(0, v^{[N_0]})>1 \quad (6.21)$$

By analogy with (4.14), for each $u\in\mathbb{Z}^d$ and for each $k\geqslant 1$ let

$$u^{[k]}=u+kv^{[N_0]}, \qquad u^{[0]}=u \quad (6.22)$$

By translation invariance $H_\zeta(u^{[k]}, u^{[k+1]})=h$ for all $k\geqslant 0$. Therefore if we write $S(u, k)$ for the collection of SDW's which start at u, end at $u^{[k]}$, contain the points $\{u, u^{[1]},..., u^{[k]}\}$, and take a step in the -1 direction immediately before and after visiting each site $u^{[j]}$ for $0\leqslant j\leqslant k$, we have the identity

$$\sum_{\theta\in S(u,k)} \zeta^\theta=h^k \qquad \text{for} \quad k\geqslant 1 \quad (6.23)$$

We will say that $S(u, k)$ and $S(w, l)$ are *disjoint* if any walks $\theta\in S(u, k)$ and $\phi\in S(w, l)$ are non-intersecting. Because of the strong constraint that at least one of every two consecutive steps must be in the -1 coordinate direction, a SDW θ that begins at a site u always stays within a cone whose apex is at u— in particular, it satisfies $|\theta(n)_j-u_j|\leqslant u_1-\theta(n)_1$ for all $j=2,..., d$. Therefore if θ and ϕ are SDW's which begin at sites $u, w\in\mathbb{Z}^d$ with $u_1=w_1$ and $|u_j-w_j|\geqslant 2N$ for some $j=2,..., d$, the walks will not intersect during their first N steps in the -1 coordinate direction. Hence for any vectors $u, w\in\mathbb{Z}^d$ with $u_1=w_1$ and $u\neq w$, it follows that $S(2N_0u, k)$ and $S(2N_0w, l)$ are disjoint for any integers k and l. Indeed, $S(2N_0u^{[k]}, 1)$ and

$S(2N_0 w^{[l]}, 1)$ are obviously disjoint for $k \neq l$. Furthermore, by the above observation and the fact that $u^{[k]} \neq w^{[k]}$, $S(2N_0 u^{[k]}, 1)$ and $S(2N_0 w^{[k]}, 1)$ are disjoint for all k. Since all walks in $S(2N_0 u, k)$ and $S(2N_0 w, l)$ are obtained by concatenation of such walks, the claim follows.

For the dimensions of the region Λ we now take $L_1 = 2lN_0$, $L_j = 8lN_0$ for $j = 2,..., d$, where l is an integer. If a vector $u \in \mathbb{Z}^d$ satisfies $u_1 = l$, and $|u_j| \leqslant 2l$ for $j = 2,..., d$, then all walks in $S(2N_0 u, 4l)$ begin and end on $\partial\Lambda$ and are contained in Λ, and are therefore MSDW's. There are $(4l+1)^{d-1}$ such vectors $u \in \mathbb{Z}^d$. Let U be the collection of these vectors. Let σ be a collection of MSDW's, obtained by choosing one MSDW from each set $S(2N_0 u, 4l)$, for all $u \in U$, and let $\mathscr{S}_U$ denote the collection of all such sets σ. Lemma 5.5 implies that each $\sigma \in \mathscr{S}_U$ is the dual of a unique configuration of SAW's Ω_σ. Also since the walks in σ are disjoint, we have $V_\sigma(\langle x, x - e_1\rangle) \leqslant 1$ for all x, and $V_{\Omega_\sigma} = V_{\sigma + E_1}$, which imply that $S_{1-}(V_{\Omega_\sigma}) = 0$. Therefore if we restrict to these configurations in (6.10), we get the lower bound

$$(z_{1+})^{-N_1(\Lambda)}\, Z(\Lambda) \geqslant \sum_{\sigma \in \mathscr{S}_U} \prod_{\theta \in \sigma} \zeta^\theta = h^{4l\,|U|} \geqslant h^{4(4^{d-1})\, l^d} \tag{6.24}$$

Since

$$|\Lambda| = (4lN_0 + 1)(16lN_0 + 1)^{d-1} \tag{6.25}$$

this implies that there is $C > 0$ such that

$$(z_{1+})^{-N_1(\Lambda)}\, Z(\Lambda) \geqslant \exp(C\,|\Lambda|) \tag{6.26}$$

Taking the log of both sides and dividing by the volume proves that $f(z) < -\log(z_{1+})$. ∎

The proof of Proposition 3.3 (iii), (iv) relies on showing that in the frozen phase typical dual flux configurations do not penetrate into the interior of Λ. We first use Lemma 5.4(a) to bound the contribution from dual flux configurations that contain a given site x. As usual, we denote by W_Φ the dual flux field associated to a flux configuration Φ, and by W_ψ the vector field associated to a SDW ψ. For $x \in \Lambda$, define the restricted partition function

$$\tilde{Z}(\Lambda; x) = \sum_{\Omega:\, x \in \operatorname{supp}(W_\Omega)} \prod_{\omega \in \Omega} z^\omega \tag{6.27}$$

where the sum runs over collections of disjoint SAW's which begin and end on $\partial\Lambda$, and whose dual flux fields are nonzero at x. By Lemma 5.4(a), for each Ω which appears on the right side of (6.27) there is a SDW ψ such that $x \in \psi$, and such that $W_\psi \leqslant W_\Omega$ and $W_\Omega - W_\psi$ is a dual flux field. If we write $W'_\Omega = W_\Omega - W_\psi$, then by imitating the derivation of (6.7), and recalling (6.4), (6.8) and (6.9), we can write (6.27) as follows:

$$\tilde{Z}(\Lambda; z) = (z_{1+})^{N_1(\Lambda)} \sum_{\Omega:\, x \in \operatorname{supp}(W_\Omega)} (z_{1+} z_{1-})^{S_{1-}(V_\Omega)} \zeta^\psi \times (z_{1+}^{-1})^{S_{1-}(W'_\Omega)} \prod_{i=2}^{d} z_{i+}^{S_{i+}(W'_\Omega)} z_{i-}^{S_{i-}(W'_\Omega)} \tag{6.28}$$

Given a SDW ψ, let $S(\psi)$ denote the following family of flux configurations:

$$S(\psi) = \{\Omega : \Omega \text{ has no loops, } W_\psi \leqslant W_\Omega = V_\Omega - E_1,\ V_\Omega - W_\psi \text{ is a flux field}\}$$

By Corollary 5.3, to each $\Omega \in S(\psi)$ there corresponds a unique flux configuration Φ such that $V_\Phi = V_\Omega - W_\psi$. Note that Φ may contain loops, although Ω does not. We let $C(\psi)$ denote this collection of flux configurations Φ, for all $\Omega \in S(\psi)$. Define

$$\tilde{Z}(\Lambda; \psi) = \sum_{\Phi \in C(\psi)} \prod_{\phi \in \Phi} z^\phi \tag{6.29}$$

Again by imitating the arguments leading to (6.7), we can rewrite (6.29) as follows:

$$\tilde{Z}(\Lambda; \psi) = (z_{1+})^{N_1(\Lambda)} \sum_{\Phi \in C(\psi)} (z_{1+} z_{1-})^{S_{1-}(V_\Phi)} (z_{1+}^{-1})^{S_{1-}(W_\Phi)} \times \prod_{i=2}^{d} z_{i+}^{S_{i+}(W_\Phi)} z_{i-}^{S_{i-}(W_\Phi)} \tag{6.30}$$

Returning to (6.28), we note that the flux configuration whose associated dual flux field is $W'_\Omega = W_\Omega - W_\psi$, belongs to $C(\psi)$. This flux configuration is uniquely determined by Ω and ψ. Conversely, a flux configuration Ω without loops may correspond to different pairs (ψ, Φ), where ψ is a SDW containing x, and $\Phi \in C(\psi)$. Therefore we obtain an upper bound for (6.28) by replacing the sum over Ω by a sum over SDW's ψ containing x, followed by a sum over all $\Phi \in C(\psi)$. The summand is almost the right side of (6.30), except that the factor $(z_{1+} z_{1-})^{S_{1-}(V_\Omega)}$ is present instead of $(z_{1+} z_{1-})^{S_{1-}(V_\Phi)} = (z_{1+} z_{1-})^{S_{1-}(V_\Omega - W_\psi)}$. However since ψ is a SDW, and therefore takes no steps in the positive first coordinate direction, we have

$S_{1-}(V_\Omega)=S_{1-}(V_\Phi)-S_{1-}(W_\psi)$, and so we obtain the following estimate for (6.28) (recall the definition of α_i in (3.9)):

$$\tilde{Z}(\Lambda, x) \leqslant \sum_{\psi:\, x\in\psi} \zeta^\psi \alpha_1^{2S_{1-}(W_\psi)} \tilde{Z}(\Lambda, \psi) \tag{6.31}$$

For a SDW θ, define

$$H(\theta)=\{x\in\theta : x+e_1\in\theta\} \tag{6.32}$$

So $|H(\theta)|=S_{1-}(W_\theta)=N_{1-}(\theta)$ counts the number of steps taken by the walk θ in the first coordinate direction. Recall the definition of $\gamma_i(z)$ in (3.10), and $\bar{z}$ in (3.4).

Lemma 6.1. Assume $\chi(\bar{z})<\infty$. Then for any SDW ψ,

$$\tilde{Z}(\Lambda; \psi) \leqslant \gamma_1(z)^{|H(\psi)|}\, Z(\Lambda) \tag{6.33}$$

Proof. Each configuration Φ in $C(\psi)$ is a collection of disjoint SAW's and loops. Since $V_\Phi+W_\psi=V_\Omega$, and there are no loops in Ω, every loop in Φ must contain at least two consecutive points which also belong to ψ. Furthermore the condition $W_\psi\leqslant W_\Omega=V_\Omega-E_1$, together with the fact that V_Ω is a flux field and hence cannot exceed 1 on any bond, implies that if $W_\psi(b)=1$ for some bond $b=\langle x, x\pm e_j\rangle$ with $j\neq 1$, then $V_\Phi(b)=0$. Hence any loop in Φ must contain two consecutive points $(x, x+e_1)$, both of which belong to ψ. Therefore any loop l in Φ can be represented by a based loop $(l(0), l(1),..., l(N))$ with $l(0)=l(N)=x$, and $l(1)=x+e_1$. Let ω be the SAW $(l(1), l(2),..., l(N))$. Then we can rewrite the weight of the loop z^l using the symmetrized weights $\bar{z}$, as follows:

$$z^l=\bar{z}^l=\sqrt{z_{1+}z_{1-}}\;\bar{z}^\omega \tag{6.34}$$

We will now derive an upper bound for (6.29). We do this by summing over all collections of disjoint loops that contain points in $H(\psi)$ (see (6.32)), and then summing over all configurations with no loops. Since we have dropped the constraint that the loops should be disjoint from the SAW's, this gives an upper bound for the sum over $\Phi\in C(\psi)$. The sum over SAW's gives the partition function. Since we assume that $\chi(\bar{z})<\infty$, this implies that $z_{1+}z_{1-}\leqslant 1$ (see Remark (iii) at the end of Section 3), and hence we can drop the interaction terms between the loops and the remaining walks.

We will write L to denote a collection of disjoint loops $\{l\}$, such that for each $l \in L$ we have $l(0) = x$ and $l(1) = x + e_1$ for some $x \in H(\psi)$. Then we have the bound

$$\tilde{Z}(\Lambda; \psi) \leqslant Z(\Lambda) \sum_{L} \prod_{l \in L} z^l \tag{6.35}$$

For all x, we have from (6.34) the estimate

$$\sum_{l:\, l(0) = x,\, l(1) = x + e_1} z^l \leqslant \sqrt{z_{1+} z_{1-}}\, \chi(\bar{z}) \tag{6.36}$$

For each $x \in H(\psi)$, let L_x be the collection of all loops satisfying $l(0) = x$ and $l(1) = x + e_1$. Then the sum in (6.35) can be estimated from above as follows:

$$\tilde{Z}(\Lambda; \psi) \leqslant Z(\Lambda) \sum_{B \subset H(\psi)} \prod_{x \in B} \prod_{l \in L_x} z^l$$

Using (6.36) this gives the bound

$$\tilde{Z}(\Lambda; \psi) \leqslant Z(\Lambda) \prod_{x \in H(\psi)} (1 + \sqrt{z_{1+} z_{1-}}\, \chi(\bar{z}))$$

and this immediately gives (6.33). ∎

We assume from now on that $\chi(\bar{z}) < \infty$, which implies that $\alpha_1 = 1$ (see Remark (iii) at the end of Section 3). Returning to (6.31), Lemma 6.1 provides the bound

$$\begin{aligned} Z(\Lambda)^{-1}\, \tilde{Z}(\Lambda; x) &\leqslant \sum_{\psi:\, x \in \psi} \zeta^{\psi} \gamma_1(z)^{|H(\psi)|} \\ &= \sum_{\psi:\, x \in \psi} [\zeta_{1-} \gamma_1(z)]^{N_{1-}(\psi)} \prod_{i=2}^{d} \zeta_{i+}^{N_{i+}(\psi)}\, \zeta_{i-}^{N_{i-}(\psi)} \end{aligned} \tag{6.37}$$

where we used $|H(\psi)| = N_{1-}(\psi)$. The factor $\gamma_1(z)$ changes the weight of each step in the first coordinate direction, and therefore the condition for finiteness of the susceptibility becomes $\lambda_1^*(z)\, \gamma_1(z) < 1$. Define

$$\tilde{d}(x; \Lambda) = \min\{|H(\psi)| : \psi \ni x,\ \psi \text{ is a SDW}\} \tag{6.38}$$

where the minimum runs over SDW's. Then for $\lambda_1^*(z)\, \gamma_1(z) < 1$, it follows from (6.37) and (6.38) that

$$Z(\Lambda)^{-1}\, \tilde{Z}(\Lambda; x) \leqslant C[\lambda_1^*(z)\, \gamma_1(z)]^{\tilde{d}(x; \Lambda)} \qquad \text{with} \quad C = (1 - \lambda_1^*(z)\, \gamma_1(z))^{-1} \tag{6.39}$$

Proof of Proposition 3.3(iii). If $\lambda_1^*(z)\,\gamma_1(z)<1$, then for all $k\geqslant 1$, and all $x_1,..., x_k\in\mathbb{Z}^d$,

$$S(x_1,..., x_k)=1$$

Proof. We will derive an upper bound for $1-S_\Lambda(x_1,..., x_k)$ which goes to zero as $\Lambda\uparrow\mathbb{Z}^d$. For a flux configuration Φ, if $x\notin\operatorname{supp}(V_\Phi)$, then $x\in\operatorname{supp}(W_\Phi)$, where W_Φ is the dual flux field. Hence

$$\sum_{\Omega:\, x\notin\operatorname{supp}(V_\Omega)}\ \prod_{\omega\in\Omega} z^\omega\leqslant\tilde{Z}(\Lambda;\, x) \tag{6.40}$$

Furthermore, since $Z(\Lambda)\,S_\Lambda(x_1,..., x_k)$ is the sum over all configurations which contain the points $\{x_1,..., x_k\}$, we have

$$Z(\Lambda)-Z(\Lambda)\,S_\Lambda(x_1,..., x_k)\leqslant\sum_{j=1}^{k}\ \sum_{\Omega:\, x_j\notin\operatorname{supp}(V_\Omega)}\ \prod_{\omega\in\Omega} z^\omega \tag{6.41}$$

Using (6.39) and (6.40) this gives

$$1-S_\Lambda(x_1,..., x_k)\leqslant C\sum_{j=1}^{k}\,(\lambda_1^*(z)\,\gamma_1(z))^{\tilde{d}(x_j;\,\Lambda)} \tag{6.42}$$

Since $\tilde{d}(x_j;\,\Lambda)\geqslant\operatorname{dist}(x,\partial\Lambda)\to\infty$ as $\Lambda\uparrow\mathbb{Z}^d$, this proves the result. ∎

In order to prove Proposition 3.3(iv) we first establish a result about the type of SAW's that occur in the frozen phase. Recall definition (5.2). For any $y\in\partial\Lambda_{1-}$, define the SAW

$$\eta_y=(y,\, y+e_1,\, y+2e_1,..., y+2L_1e_1) \tag{6.43}$$

(recall that $\Lambda=[-L_1, L_1]\times\cdots\times[-L_d, L_d]$).

Lemma 6.2. For $y\in\partial\Lambda_{1-}$, let $y_*=\min(2L_1, |y_2-L_2|, |y_2+L_2|,..., |y_d-L_d|, |y_d+L_d|)$. If $\lambda_1^*(z)\,\gamma_1(z)<1$, there is $C<\infty$ such that for all $y\in\partial\Lambda_{1-}$,

$$\sum_{\Omega:\,\eta_y\notin\Omega}\ \prod_{\omega\in\Omega} z^\omega\leqslant C(2L_1)[\lambda_1^*(z)\,\gamma_1(z)]^{y_*}\,Z(\Lambda) \tag{6.44}$$

Proof. Suppose Ω is a collection of SAW's which does not contain η_y. Then there must be some j satisfying $0\leqslant j\leqslant 2L_1-1$ such that

$y+je_1,\ y+(j+1)\,e_1\notin\omega$ for any $\omega\in\Omega$, and hence $y+je_1\in\operatorname{supp}(W_\Omega)$. Therefore using (6.27),

$$\sum_{\Omega:\,\eta_y\notin\Omega}\ \prod_{\omega\in\Omega} z^\omega \leqslant \sum_{j=0}^{2L_1-1} \tilde{Z}(\Lambda;\, y+je_1) \tag{6.45}$$

Using (6.39) gives the following estimate for (6.45):

$$\begin{aligned}\sum_{\Omega:\,\eta_y\notin\Omega}\ \prod_{\omega\in\Omega} z^\omega &\leqslant C \sum_{j=0}^{2L_1-1} [\lambda_1^*(z)\,\gamma_1(z)]^{\tilde{d}(y+je_1;\,\Lambda)}\, Z(\Lambda)\\ &\leqslant C(2L_1)[\lambda_1^*(z)\,\gamma_1(z)]^{\tilde{d}(y+j_*e_1;\,\Lambda)}\, Z(\Lambda)\end{aligned} \tag{6.46}$$

where j_* is defined by

$$\tilde{d}(y+j_*e_1;\Lambda)=\min\{\tilde{d}(y+je_1;\Lambda):0\leqslant j\leqslant 2L_1-1\} \tag{6.47}$$

A simple argument shows that

$$\tilde{d}(y+j_*e_1;\Lambda)\geqslant \min(2L_1,\, |y_2-L_2|,\, |y_2+L_2|,\ldots,\, |y_d-L_d|,\, |y_d+L_d|) \tag{6.48}$$

and this establishes the result. ∎

Proof of Proposition 3.3(iv). If $\lambda_1^*(z)\,\gamma_1(z)<1$, and condition (2.10) holds, then for all $k\geqslant 1$, and all $x_1,\ldots, x_k\in\mathbb{Z}^d$,

$$\tau(x_1,\ldots, x_k)=\begin{cases}1 & \text{if } x_1,\ldots, x_k \text{ lie on a straight line in the 1st direction}\\ 0 & \text{otherwise}\end{cases}$$

Proof. Suppose first that $x_1,\ldots, x_k$ lie on a straight line in the first coordinate direction. Let $y\in\partial\Lambda_{1-}$ such that $x_1,\ldots, x_k\in\eta_y$. Then

$$\tau_\Lambda(x_1,\ldots, x_k)\geqslant Z(\Lambda)^{-1}\sum_{\Omega:\,\eta_y\in\Omega}\ \prod_{\omega\in\Omega} z^\omega \tag{6.49}$$

Using the result of Lemma 6.2, and observing that $y_*\to\infty$ as $\Lambda\uparrow\mathbb{Z}^d$ (faster than $\log L_1$ by condition (2.10)) immediately gives the result.

Suppose now that $x_1,\ldots, x_k$ do not lie on a straight line in the first coordinate direction. Let $y_1,\ldots, y_k\in\partial\Lambda_{1-}$ such that $x_j\in\eta_{y_j}$ for $j=1,\ldots, k$. Then

$$\tau_\Lambda(x_1,\ldots, x_k)\leqslant Z(\Lambda)^{-1}\sum_{j=1}^{k}\ \sum_{\Omega:\,\eta_{y_j}\notin\Omega}\ \prod_{\omega\in\Omega} z^\omega \tag{6.50}$$

Again Lemma 6.2 with condition (2.10) gives the desired result. ∎

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